Math 31: Abstract Algebra
Fall 2018 - Homework 7

Return date: Wednesday 10/31/18

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keywords: *homomorphism theorem, graph automorphisms*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 1. *(4 points)* Let $f : A \to C$ and $g : B \to D$ be two functions.
Let $f \times g : A \times B \to C \times D$ be function defined by

$$(f \times g)(a, b) := (f(a), g(b)) \text{ for all } (a, b) \in A \times B.$$

Show that $f \times g$ bijective $\Leftrightarrow$ f and g bijective.

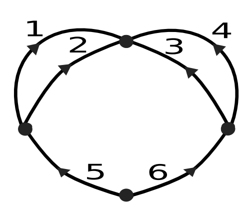
exercise 2. *(6 points)* Let $(G, \cdot)$ and $(H, \cdot)$ be two groups. Let furthermore $N \lhd G$ be a normal subgroup of G and $M \lhd H$ be a normal subgroup of H.

a) Show that the function $f : G \times H \to (G/N) \times (H/M), (a, b) \mapsto f(a, b) := (Na, Mb)$ is a surjective homomorphism.

b) Find the kernel $\ker(f)$ of f.

c) Use the homomorphism theorem to conclude that $(G \times H)/(N \times M) \simeq (G/N) \times (H/M)$.

exercise 3. *(7 points)* Let Γ be the following graph with edges $E = \{1, 2, 3, 4, 5, 6\}$.



a) Show that $\mathrm{Aut}(\Gamma)$ is (isomorphic to) a subgroup of $(S_4, \circ)$.

b) Show that $\mathrm{Aut}(\Gamma)$ contains an element r of order 4 and find $\#\,\mathrm{Aut}(\Gamma)$.

c) Show that $\mathrm{Aut}(\Gamma)$ is a non-abelian group.

d) Draw the Cayley graph of $(\mathrm{Aut}(\Gamma), \circ)$ with generators r and s, where s is an element of order 2. Then look up the subgroups of S_4 and decide to which group $\mathrm{Aut}(\Gamma)$ is isomorphic.

exercise 4. *(3 points)* Let $\Gamma := \Gamma(\mathbb{Z}_5, \{1\})$ be the Cayley graph of $\mathbb{Z}_5$ generated by $\{1\}$. Show that $\mathrm{Aut}(\Gamma) \simeq (\mathbb{Z}_5, +_5)$.
Note: In general $\mathrm{Aut}(\Gamma(\mathbb{Z}_n, \{1\})) \simeq (\mathbb{Z}_n, +_n)$.
