

Math 31: Abstract Algebra
Fall 2018 - Homework 6

Return date: Wednesday 10/24/18

keywords: *quotient groups, fundamental homomorphism theorem*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 1. (8 points) Let $(\mathbb{R}^3, +)$ be the group \mathbb{R}^3 , where $+$ denotes the vector addition. Let

$$N = \{(x, y, z) \in \mathbb{R}^3, y = 2x\}.$$

- a) Show that N is a subgroup. Why is N normal?
- b) In geometrical terms, describe N and the elements of the quotient group \mathbb{R}^3/N .
- c) In geometrical terms or otherwise, describe the operation of \mathbb{R}^3/N .
- d) Let $f : (\mathbb{R}^3, +) \rightarrow (\mathbb{R}, +)$ be the homomorphism given by

$$f(x, y, z) := (4, -2, 0) \bullet (x, y, z).$$

Show that f is surjective and that $N = \ker(f)$. Then apply the fundamental homomorphism theorem to f .

Note: You do **NOT** have to show that f is a homomorphism.

exercise 2. (8 points) Let (G, \cdot) be a group and $N \triangleleft G$ be a normal subgroup and G/N be the corresponding quotient. Show that:

- a) For $a \in G, Na \in G/N$: $\text{ord}(Na)$ is a factor of $\text{ord}(a)$.
- b) If the index $(G : N) = m$. Then the order of every element of G/N is a factor of m .
- c) If every element of N has finite order and every element of G/N has finite order, then every element of G has finite order.
- d) If G is a cyclic group then G/N is a cyclic group.
- e) Let G be an abelian group. If every element in G/N has a square root and every element in N has a square root, then every element in G has a square root.

Note: An element h in a group (H, \cdot) has a square root, if there is an element $x \in H$, such that $h = x^2$.

exercise 3. (4 points) Consider $(\mathbb{Z}_n, +_n)$. If m is a factor of n , then by **Ch.11, B4**, there is a cyclic subgroup $\langle k \rangle$ of order m in $(\mathbb{Z}_n, +_n)$. Show that the quotient group $\mathbb{Z}_n/\langle k \rangle$ is a cyclic group of order $\frac{n}{m}$.

Give a counterexample that shows that in general $(\mathbb{Z}_n/\langle k \rangle) \times \langle k \rangle$ is not isomorphic to $(\mathbb{Z}_n, +_n)$.
