Math 31: Abstract Algebra
Fall 2018

10/24/18

Graph morphisms

Question: How can we understand a graph morphism intuitively?

Idea: The key to understanding a graph morphism is that the condition

$$\tilde{\delta}(f_E(e)) = f_V \times f_V(\delta(e)) \quad \text{for all} \quad e \in E$$

can be interpreted as: if an edge e goes to $f(e)$ then its endpoints must follow or also if a vertex v goes to $f(v)$ then its attached edges must follow. Intuitively a map $f = (f_V, f_E)$ is a graph morphism if it "folds" the graph Γ into $\tilde{\Gamma}$ without disrupting it, such that the directions of the edges are respected.

Examples:

i) Clearly we must have $f(e_1) = f(e_2) = \tilde{e}$ as this is the only choice. Then v_1 and v_2 must follow, i.e. $f(v_1) = \tilde{v}_1$ and $f(v_2) = \tilde{v}_2$.

ii) Again we must have $f(e_1) = f(e_2) = \tilde{e}$. As f is a graph morphism we have

$$f(e_1) = \tilde{e} \Rightarrow \tilde{\delta}(f(e_1)) = \tilde{\delta}(\tilde{e}) = (\tilde{v}_1, \tilde{v}_2) = (f(v_1), f(v_2)).$$

Especially that means $f(e_1) = \tilde{e} \Rightarrow f(v_1) = \tilde{v}_1$. Similarly $f(e_2) = \tilde{e} \Rightarrow f(v_1) = \tilde{v}_2$. But $f(v_1) = \tilde{v}_1$ contradiction.

iii) Is there a graph morphism from Γ_1 to Γ_2? Explain your answer.

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Automorphism groups of graphs

Aim: The symmetries of a graph form the automorphism group of the graph. We will show that a Cayley graph has "many" symmetries. In fact it is so symmetric, that it is homogeneous, meaning that it "looks the same" from every vertex.

Theorem 1 If $\Gamma = (V, E, \delta)$ is a graph, then the set

$$\mathrm{Aut}(\Gamma) = \{f : \Gamma \to \Gamma \mid f \text{ automorphism } \}$$

together with composition $\circ$ of morphisms forms a group $(\mathrm{Aut}(\Gamma), \circ)$, the **automorphism group of** Γ.

proof By **Theorem 5.2.)** we have that $f = (f_V, f_E) : \Gamma \to \Gamma$ isomorphism if and only if f_V and f_E are bijective functions. Hence

$$f \text{ automorphism } \Rightarrow f = (f_V, f_E) \in (S_V \times S_E, \circ).$$

It remains to prove the subgroup criteria:

1.) $id_\Gamma = (id_V, id_E) \in \mathrm{Aut}(\Gamma)$.

2.) $f \in \mathrm{Aut}(\Gamma) \overset{\textbf{Theorem 5.2.)}}{\Rightarrow} f^{-1} \in \mathrm{Aut}(\Gamma)$.

3.) $f, g \in \mathrm{Aut}(\Gamma) \overset{\textbf{Theorem 5.1.,2.)}}{\Rightarrow} f \circ g$ graph morphism, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.
Hence $f \circ g \in \mathrm{Aut}(\Gamma)$.

Hence $\mathrm{Aut}(\Gamma)$ is a subgroup of $(S_V \times S_E, \circ)$ by the subgroup criteria $\square$.

Example Consider the following graph Γ with four vertices and three edges. To which group is $\mathrm{Aut}(\Gamma)$ isomorphic ?

Solution: Let $f \in \mathrm{Aut}(\Gamma)$. We see that the vertex v_4 must be fixed by f, i.e. $f(v_4) = v_4$ as its edges must follow. If $f(v_4) \neq v_4$ then we would have that f is not injective on the edges. Furthermore f is completely determined by f_V as the edges must follow the vertices. This means that f is determined by $f_V \in S_{V \setminus \{v_4\}} = S_3$. We know that $\mathrm{Aut}(\Gamma)$ is (isomorphic to) a subgroup of $(S_3, \circ)$. All permutations of the first three vertices are possible. Hence $\mathrm{Aut}(\Gamma) \simeq S_3$.