Math 31: Abstract Algebra
Fall 2018

10/22/18

Graph morphisms

Question: What makes Cayley graphs special among graphs?

Examples: Draw the Cayley graph $\Gamma(\mathbb{Z}_6, \{1\})$ and the Cayley graph $\Gamma(\mathbb{Z} \times \mathbb{Z}, \{(0,1),(1,0)\})$:

In a way Cayley graphs are the "pearls" among graphs. They are very regular. We already know that locally every vertex has the same "neighbourhood":

Lemma 1 Let $G = (G, \cdot)$ be a group with neutral element $e \in G$ and $S \subset G$ a generating set, i.e. $\langle S \rangle = G$, such that $\#S = n$ for some $n \in \mathbb{N}$. Let $\Gamma = \Gamma(G, S)$ be the corresponding Cayley graph. Then

$$\text{val}(g) = \text{val}(e) = 2n \quad \text{for all} \quad g \in G = V(\Gamma).$$

proof HW 3

Questions

1.) How can we define maps between graphs?
2.) When are two graphs equal?

To answer these questions properly we first recall the definition of the Cartesian product.

Note 2 (Cartesian product) If A and B are sets then

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

is the **Cartesian product** of A and B.
We have: $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$. For two functions $f : A \to C$ and $g : B \to D$ we define by $f \times g : A \times B \to C \times D$ the function given by

$$(a,b) \to f \times g(a,b) := (f(a), g(b)) \quad \text{for all} \quad a \in A\,,\, b \in B.$$

It follows that:

Lemma 3 f and g are both bijective $\Leftrightarrow$ $f \times g$ bijective.

proof HW 7

Example Let $A = C = (\mathbb{Z}_3, +_3)$ and $B = D = (\mathbb{Z}_4, +_4)$. Let f and g be the maps defined by

$$f(a) := a +_3 1 \quad \text{and} \quad g(b) := b +_4 2.$$

Draw a picture of $A \times B = \mathbb{Z}_3 \times \mathbb{Z}_4$ in a coordinate system on the left and a picture of $C \times D = \mathbb{Z}_3 \times \mathbb{Z}_4$ on the right and visualize the map $f \times g$.

To define a proper map $f : \Gamma \to \tilde{\Gamma}$ between graphs we must assure that the image $f(\Gamma')$ of a subgraph Γ' of Γ is a subgraph of $\tilde{\Gamma}$. Such a map is called a **graph morphism**. This condition is guaranteed if we use the following definition.

Definition 4 (Graph morphism) Let $\Gamma = (V, E, \delta)$ and $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \tilde{\delta})$ be two graphs. Then

1.) a **(graph) morphism** $f : \Gamma \to \tilde{\Gamma}$ is a pair of two maps $f = (f_V, f_E)$

- $f_V : V \to \tilde{V}$ and $f_E : E \to \tilde{E}$, such that
- for all $e \in E$ we have: if $\delta(e) = (u, w)$ then $\tilde{\delta}(f_E(e)) = (f_V(u), f_V(w))$ or shortly

$$\tilde{\delta}(f_E(e)) = f_V \times f_V(\delta(e)).$$

This means that if two vertices are connected by an edge e then the images of the vertices must be connected to $f_E(e)$ and the map f preserves starting and endpoints.

2.) f is called a **(graph) isomorphism**, if there is a graph morphism $g : \Gamma \to \tilde{\Gamma}$, such that

$$f \circ g = id_{\tilde{\Gamma}} \quad \text{and} \quad g \circ f = id_{\Gamma}.$$

An isomorphism $f : \Gamma \to \Gamma$ is called a **(graph) automorphism**.

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Note 1.) If it is clear from the context we skip the subscript in $f_V(v)$ and $f_E(e)$ and just write $f(v)$ and $f(e)$.
2.) The condition $\tilde{\delta}(f_E(e)) = f_V \times f_V(\delta(e))$ can be interpreted as: if an edge e goes to $f(e)$ then its endpoints must follow or also if a vertex v goes to $f(v)$ then its attached edges must follow.

Examples

Theorem 5 1.) The composition of two graph morphisms is again a graph morphism.
2.) $f : \Gamma \to \tilde{\Gamma}$ isomorphism $\Leftrightarrow$ f_V and f_E are both bijective.

proof of Theorem 5 1.) For $i \in \{1, 2, 3\}$ let $\Gamma_i = (V_i, E_i, \delta_i)$ be a graph and let $f = (f_{V_1}, f_{E_1}) : \Gamma_1 \to \Gamma_2$ and $g = (g_{V_2}, g_{E_2}) : \Gamma_2 \to \Gamma_3$ be two graph morphisms. Then clearly

$$g \circ f = (g_{V_2} \circ f_{V_1}, g_{E_2} \circ f_{E_1})$$

is a pair of maps, such that $g_{V_2} \circ f_{V_1} : V_1 \to V_3$ and $g_{E_2} \circ f_{E_1} : E_1 \to E_3$. It remains to show the second condition. We have for all $e_1 \in E_1, e_2 \in E_2$:

$$\delta_2(f_{E_1}(e_1)) = f_{V_1} \times f_{V_1}(\delta_1(e_1)) \quad \text{and} \quad \delta_3(g_{E_2}(e_2)) = g_{V_2} \times g_{V_2}(\delta_2(e_2))$$

Hence for $e_2 = f_{E_1}(e_1)$ we obtain first with second equation and then with the first equation above

$$\delta_3(g_{E_2} \circ f_{E_1}(e_1)) = g_{V_2} \times g_{V_2}(\delta_2(f_{E_1}(e_1))) = g_{V_2} \times g_{V_2} \circ f_{V_1} \times f_{V_1}(\delta_1(e_1)) = g_{V_2} \circ f_{V_1} \times g_{V_2} \circ f_{V_1}(\delta_1(e_1)).$$

This shows that the second condition for a morphism in **Definition 4** is also satisfied.

2.) To show the second part of **Theorem 5** we show the equivalence.
"$\Rightarrow$" We recall from **Chapter 5** that

a) $f_V : V \to \tilde{V}$ bijective $\Leftrightarrow \exists f_V^{-1} : \tilde{V} \to V$, such that

$$f_V^{-1} \circ f_V = id_V \quad \text{and} \quad f_V \circ f_V^{-1} = id_{\tilde{V}}$$

b) $f_E : E \to \tilde{E}$ bijective $\Leftrightarrow \exists f_E^{-1} : \tilde{E} \to E$, such that

$$f_E^{-1} \circ f_E = id_E \quad \text{and} \quad f_E \circ f_E^{-1} = id_{\tilde{E}}$$

As $f \circ g = id_{\tilde{V}}$ and $g \circ f = id_V$ we have that $f_V^{-1} = g_{\tilde{V}}$ and $f_E^{-1} = g_{\tilde{E}}$ and f_V and f_E are bijective.

"$\Leftarrow$" Conversely if f_V and f_E are bijective then there exist the inverse maps $f_V^{-1} : \tilde{V} \to V$ and $f_E^{-1} : \tilde{E} \to E$ and we can set $g = f^{-1} = (f_V^{-1}, f_E^{-1})$.
It remains to show that f^{-1} satisfies the second condition, i.e. for all $\tilde{e} \in \tilde{V}$ we have

$$\delta(f_E^{-1}(\tilde{e})) = f_V^{-1} \times f_V^{-1}(\tilde{\delta}(\tilde{e})). \tag{1}$$

We know that

$$\begin{aligned} \tilde{e} = f_E(e) \Leftrightarrow e &= f_E^{-1}(\tilde{e}) \quad \text{for all} \quad \tilde{e} \in \tilde{E}, e \in E & (2) \\ \tilde{v} = f_V(v) \Leftrightarrow v &= f_V^{-1}(\tilde{v}) \quad \text{for all} \quad \tilde{v} \in \tilde{V}, v \in V & (3) \\ \tilde{\delta}(f_E(e)) &= f_V \times f_V(\delta(e)). & (4) \end{aligned}$$

By the definition of $f_V \times f_V$ we also have that $(f_V \times f_V)^{-1} = f_V^{-1} \times f_V^{-1}$. To prove (1) we use (2) and set $e = f_E^{-1}(\tilde{e})$ in (1):

$$\delta(f_E^{-1}(\tilde{e})) = \delta(e) = (f_V^{-1} \times f_V^{-1}) \circ (f_V \times f_V)(\delta(e)) \overset{(4)}{=} (f_V^{-1} \times f_V^{-1})\tilde{\delta}(f_E(e)) \overset{(2)}{=} (f_V^{-1} \times f_V^{-1})\tilde{\delta}(\tilde{e})$$

This proves our statement (1). This concludes the proof of **Theorem 5**. $\square$