Math 31: Abstract Algebra
Fall 2018

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Cayley graphs

Aim: To understand how groups look like, we represent them by a graph,
the so called **Cayley graph**.

Definition 1 (Graph) A **graph** is an ordered triplet $\Gamma = (V, E, \delta)$, such that

- $V = V(\Gamma)$: set of **vertices** or **nodes** of Γ
- $E = E(\Gamma)$: set of **edges** of Γ
- $\delta : E \to V \times V = \{(u, v) \mid u, v \in V\}$: the **gluing map** that assigns to each edge an ordered pair of vertices.

If $\delta(e) = (u, v) = (o(e), t(e))$, then we call

- $u = o(e)$ the **origin** of the edge $e \in E$
- $v = t(e)$ the **terminus** of the edge $e \in E$

Note: As the edges have an origin and terminus, they are oriented. We can think of the edges as arrows, where the tip is at the terminus.

Example: Let $V = \{v_1, v_2, v_3, v_4\}$ be four vertices (points) and $E = \{e_1, e_2, e_3, e_4\}$ four edges (arrows). Draw your own graph Γ with these edges and vertices. Then use the picture to describe the gluing function δ.

Notation

- we often abbreviate (V, E, δ) by (V, E). However, we need δ as there may be multiple edges between two vertices. Otherwise we could use a simpler definition.

- two vertices u, v are called **adjacent** if they are connected by an edge. In this case we write shortly $u \sim v$.
- similarly two edges e, f are called **adjacent** if they have a common vertex. In this case we also write $e \sim f$.
- a **loop** is an edge that connects a single vertex.
- a **simple graph** is a graph that has no loops pairs of vertices connected by multiple edges.
- the **valency** $\mathrm{val}(v)$ of a vertex v is the number of edges connected to a vertex, where a loop is counted twice. Informally this is the number of "half-edges" connected to a vertex.

A subgraph is a part of a graph that is also a graph. To assure that it is indeed a graph, we must make sure that its edges are connected to vertices inside the subgraph:

Definition 2 (Subgraph) If $\Gamma = (V, E, \delta)$ is a graph, then $\Gamma' = (V', E')$ is a **subgraph** of Γ if

1.) $V' \subset V$ and $E' \subset E$

2.) for all $e' \in E$ we have that $\delta(e') \subset V' \times V'$

Example

Definition 3 (Cayley graph) If $G = (G, \cdot)$ is a group and $S \subset G$ a generating set, i.e. $\langle S \rangle = G$. Then the (directed) graph $\Gamma = (V, E, \delta)$ defined by

- $V = G$
- $E = G \times S = \{(g, s) \mid g \in G, s \in S\}$
- $\delta : G \times S \to G \times G$, $\boxed{\delta : (g, s) \mapsto \delta(g, s) = (g, g \cdot s)}$.

is the **Cayley graph** $\Gamma = \Gamma(G, S)$ of G with respect to S.

Though this is not part of the definition, we will usually restrict ourselves to the case where the generating set is finite, i.e. $\boxed{\# \mathrm{S} = \mathrm{n}}$ for some $n \in \mathbb{N}$.

Examples Here we assume that e is the neutral element of G.

i) $\Gamma_1 = \Gamma(\{e\}, \{e\})$, where $G = \{e\}$ is the group with one element.

ii) $\Gamma_2 = \Gamma(\mathbb{Z}_5, \{1\})$

iii) $\Gamma_3 = \Gamma(\mathbb{Z}, \{1\})$

iv) For $\Gamma_4 = \Gamma(G, S)$, where $S = \{s_1, s_2, s_3\}$. Draw all edges connected to the vertex e.

Exercise

v) $\Gamma_5 = \Gamma(\mathbb{Z}, \{2, 3\})$

vi) $\Gamma_5 = \Gamma(\mathbb{Z}_5, \{0, 1, 2\})$