NAME: ______________________________

Math 31

Midterm Examination

Rules

- This is a **closed book exam**. No document is allowed.
- Cell phones and other electronic devices must be turned off.
- Questions and requests for clarification can be addressed to the instructor only.
- You are allowed to use the result of a previous question even if you did not prove it, as long as you indicate it explicitly.

Grading

- In order to receive full credit, solutions must be **justified with full sentences**.
- The clarity of your explanations will enter into the appreciation of your work.

Last piece of advice

Read the entire exam before you start to write anything.

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	Total
Points	6	6	7	6	6	6	7	6	50
Score									

1. *(6 points)* Let $*$ be the operation on $\mathbb{R}$ given by

$$x * y = 2^x \cdot 2^y.$$

Explain whether or not

i) the operation is commutative,

ii) there is an identity element e with respect to $*$,

iii) if for every element there is an inverse with respect to $*$.

2. *(6 points)* Let $G = \{e, a, b, c\}$ be a set of four elements, where e denotes the neutral element.
Using an operation table, find all possible groups with these four elements, where besides e, ONLY a is its own inverse. **Justify your answer**.

3. *(7 points)* Let $(G, \cdot)$ be a group.

a. Is $f : G \to G, x \mapsto f(x) = x^{-1}$ a bijective function? Justify your answer.

b. Show that for fixed $a \in G$ the function $h : G \to G, x \mapsto h(x) = axa^{-1}$ is a bijective function.

c. Is the function $h : G \to G$ from part **b.** a group isomorphism? Justify your answer.

4. *(6 points)* Recall that a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with real coefficients is invertible if and only if its determinant

$$\det(A) = ad - bc \neq 0.$$

The multiplicative group of invertible matrices of size 2×2 is denoted by $\mathrm{GL}_2(\mathbb{R})$.

a. Let $(\mathbb{R}^*, \cdot)$ be the group of real numbers without zero and multiplication as operation. Consider the subgroup $H \leq \mathbb{R}^*$ given by

$$H := \{2^n, n \in \mathbb{Z}\}.$$

Prove that the subset

$$M := \{A \in \mathrm{GL}_2(\mathbb{R}), \det(A) \in H\}$$

is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

b. Is the map $\det : M \to H$ injective? Justify your answer.

5. *(6 points)* **Prove or disprove** the following statements about groups.
Note: To disprove a statement an counterexample is sufficient.

a. Let $(G, \cdot)$ be a group. If for all a, b in G: $(ab)^2 = a^2b^2$, then G is abelian.

b. If every element of a group $(G, \cdot)$ is its own inverse, then G is abelian.

c. If $(G, \cdot)$ and $(H, *)$ are groups, such that $\langle a \rangle = G$ and $\langle b \rangle = H$. Then the product group $G \times H$ can be generated by a single element.

6. *(6 points)* In the set of real numbers $\mathbb{R}$, let $\sim$ be the relation given by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}\,.$$

a. Show that $\sim$ is an equivalence relation in $\mathbb{R}$.

b. Write down the two equivalence classes

$$[1] \text{ of } 1 \in \mathbb{R} \quad \text{and} \quad [\sqrt{2}] \text{ of } \sqrt{2} \in \mathbb{R}\,.$$

7. *(7 points)* Consider the group $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, +_2 \times +_2 \times +_2)$.

a. List all elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

b. Draw the Cayleygraph

$$\Gamma_1 = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \{(1,0,0), (0,1,0), (0,0,1)\})$$

of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with respect to the generating set $\{(1,0,0), (0,1,0), (0,0,1)\}$.

8. *(6 points)* Let $(G, \cdot)$ be a group. Let $H \leq G$ be a subgroup of G and $a \in G$ be a fixed element. Set

$$K := aHa^{-1} = \{aha^{-1}, \text{ such that } h \in H\}.$$

Show that K is a subgroup of G.