NAME: GRADE:

Math 31

Midterm Exam II

Rules

- This is a **closed book exam**. No document is allowed.
- Cell phones and other electronic devices must be turned off.
- Questions and requests for clarification can be addressed to the instructor only.
- You are allowed to use the result of a previous question even if you did not prove it, as long as you indicate it explicitly.

Grading

- In order to receive full credit, solutions must be **justified with full sentences**.
- The clarity of your explanations will enter into the appreciation of your work.

Last piece of advice

Read the entire exam before you start to write anything.

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	Total
Points	6	6	6	7	5	5	8	7	50
Score									

1. *(6 points)* Recall that a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real coefficients is invertible if and only if $\det(A) = ad - bc \neq 0$. The multiplicative group of invertible matrices of size 2×2 is denoted by $(\mathrm{GL}_2(\mathbb{R}), \cdot)$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \frac{1}{3} \\ 3 & 0 \end{pmatrix}$$

be two elements in $\mathrm{GL}_2(\mathbb{R})$.

a. Determine the order of A and the order of B.

b. Show that AB has infinite order.

2. *(6 points)* Let $p \in (S_9, \circ)$ be the permutation

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 8 & 2 & 4 & 1 & 9 & 6 & 5 \end{pmatrix}.$$

a. Write p as a product of disjoint cycles.

b. Calculate p^{22}.

c. Consider the elements in $(S_5, \circ)$. Into how many disjoint cycles can an element q of S_5 maximally decompose? Justify your answer.
Note: A cycle must have at least two elements.

d. What is the maximal order an element in $(S_5, \circ)$ can have? Justify your answer and give an example.

3. *(6 points)* Let $(G, \cdot)$ be a group with neutral element e. **Prove or disprove** the following statements.
Note: To disprove a statement an counterexample is sufficient.

a. G and $\{e\}$ are normal subgroups in G.

b. Let $N \lhd G$ be a normal subgroup. Then any subgroup of N is normal in G.

c. Suppose H and K are subgroups of G, such that $H \neq K$. If $\#H = \#K = p$, where p is a prime number, then $H \cap K = \{e\}$. **Hint:** Lagrange's theorem.

4. *(7 points)* Let $(\mathrm{GL}_2(\mathbb{R}), \cdot)$ be the group of invertible 2×2 matrices.

a. Show that

$$f : (\mathrm{GL}_2(\mathbb{R}), \cdot) \to (\mathbb{R}, +), A \mapsto f(A) := \ln(|\det(A)|)$$

is a group homomorphism.

b. Find the kernel $\ker(f)$ of f.

c. Is f surjective? Justify your answer.

d. Apply the fundamental homomorphism theorem (FHT) to f.

5. *(5 points)* Let $(G, \cdot)$ be a group. Its *center* is by definition the subgroup C of elements that commute with all the elements of G:

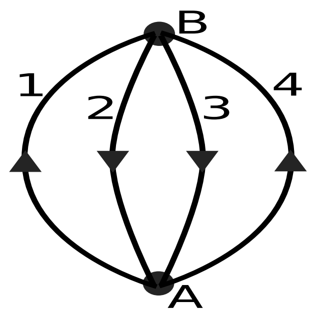
$$C = \{c \in G\ ,\ cx = xc \quad \text{for all } x \in G\} \leq G.$$

a. Show that C is a normal subgroup of G.

b. Show that if G/C is cyclic, then G is abelian.
Hint: Let $G/C = \langle Ca \rangle$ be generated by the coset Ca for some $a \in G$.

6. *(5 points)* Let $(G, \cdot)$ be a group and $N \lhd G$ be a normal subgroup. Suppose that the order of every element in N and in G/N is a power of 5.
Show that the order of every element in G is a power of 5.

7. *(8 points)* Let Γ be the following graph with vertices $V = \{A, B\}$ and edges $E = \{1, 2, 3, 4\}$.



a. List all elements of its automorphism group $\mathrm{Aut}(\Gamma)$.

b. Draw the Cayley graph of $(\mathrm{Aut}(\Gamma), \circ)$.
Hint: $\mathrm{Aut}(\Gamma)$ can be generated by three elements.

8. *(7 points)* Let $(G, \cdot)$ be a group. Let $N \lhd G$ and $L \lhd G$ be normal subgroups of G, such that $\boxed{N \subseteq L}$. On the quotient groups we define

$$f : G/N \to G/L, Na \mapsto f(Na) = La.$$

a. Show that f is well-defined, i.e. if $Na = Nb$ in G/N then $f(Na) = f(Nb)$ in G/L.

b. Show that f is a group homomorphism.

c. Find the kernel $ker(f)$ of f.

d. Conclude that $(G/N)/(L/N)$ is isomorphic to G/L.