Math 31: Abstract Algebra
Fall 2017 - Homework 6

Return date: Wednesday 10/25/17

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keywords: *quotient groups, fundamental homomorphism theorem*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 1. *(8 points)* Let $(\mathbb{R}^3, +)$ be the group $\mathbb{R}^3$, where $+$ denotes the vector addition. Let

$$N = \left\{(x, y, z) \in \mathbb{R}^3, y = 2x\right\}.$$

a) Show that N is a subgroup. Why is N normal?

b) In geometrical terms, describe N and the elements of the quotient group $\mathbb{R}^3/N$.

c) In geometrical terms or otherwise, describe the operation of $\mathbb{R}^3/N$.

d) Let $f : (\mathbb{R}^3, +) \to (\mathbb{R}, +)$ be the homomorphism given by

$$f(x, y, z) := (4, -2, 0) \bullet (x, y, z).$$

Show that f is surjective and that $N = \ker(f)$. Then apply the fundamental homomorphism theorem to f.
Note: You do **NOT** have to show that f is a homomorphism.

exercise 2. *(8 points)* Let $(G, \cdot)$ be a group and $N \lhd G$ be a normal subgroup and G/N be the corresponding quotient. Show that:

a) For $a \in G, Na \in G/N$: $\operatorname{ord}(Na)$ is a factor of $\operatorname{ord}(a)$.

b) If the index $(G : N) = m$. Then the order of every element of G/N is a factor of m.

c) If every element of N has finite order and every element of G/N has finite order, then every element of G has finite order.

d) If G is a cyclic group then G/N is a cyclic group.

e) Let G be an abelian group. If every element in G/N has a square root and every element in N has a square root, then every element in G has a square root.
Note: An element h in a group $(H, \cdot)$ has a square root, if there is an element $x \in H$, such that $h = x^2$.

exercise 3. *(4 points)* Consider $(\mathbb{Z}_n, +_n)$. If m is a factor of n, then by **Ch.11, B4**, there is a cyclic subgroup $\langle k\rangle$ of order m in $(\mathbb{Z}_n, +_n)$. Show that the quotient group $\mathbb{Z}_n/\langle k\rangle$ is a cyclic group of order $\frac{n}{m}$.
Give a counterexample that shows that in general $(\mathbb{Z}_n/\langle k\rangle) \times \langle k\rangle$ is not isomorphic to $(\mathbb{Z}_n, +_n)$.
