Math 31: Abstract Algebra
Fall 2017 - Homework 3

Return date: Wednesday 10/04/17

keyword: *isomorphisms, Cayley graphs, equivalence relations*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 1. *(5 points)* Isomorphic and non-isomorphic groups.

a) Let $(E,+) = (2\,\mathbb{Z},+)$ be the group of even integers. Show that E is isomorphic to $\mathbb{Z}$. In general, is $(n\,\mathbb{Z},+)$ isomorphic to $(\mathbb{Z},+)$?

b) Show that $(\mathbb{R},+)$ is not isomorphic to $(\mathbb{R}^*,\cdot)$, where $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$.
Hint: Look at the properties of -1 in $\mathbb{R}^*$. Does $(\mathbb{R},+)$ have such an element?

exercise 2. *(6 points)* Consider the groups $(\mathbb{Z}_2\times\mathbb{Z}_3, +_2\times +_3)$ and $(S_3,\circ)$, where $S_3 = S(\{1,2,3\})$. Recall that S_3 consists of the elements

$$id\ ,\ s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, s_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, r_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \text{ and } r_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

a) Use the operation table for $\mathbb{Z}_2\times\mathbb{Z}_3$ from **Homework 2** to draw the Cayley graph $\Gamma_1 = \Gamma(\mathbb{Z}_2\times\mathbb{Z}_3, \{(1,0),(0,1)\})$.

b) Write the operation table for $(S_3,\circ)$.

c) Draw the Cayley graph $\Gamma_2 = \Gamma(S_3,\{s_1,r_1\})$ and compare it with Γ_1.

exercise 3. *(3 points)* Let $(G,\cdot)$ be a group with neutral element e. Let S be a generating set, i.e. $G = \langle S\rangle$ consisting of n elements. Let $\Gamma(G,S)$ be the corresponding Cayley graph. Show that

$$\mathrm{val}(h) = \mathrm{val}(e) = 2n \ \text{ for all } \ h \in G.$$

exercise 4. *(6 points)* Prove that each of the following is an equivalence relation on the indicated set. Then describe the partition associated with the equivalence relation.

a) In $\mathbb{Q}$: $q\sim r \Leftrightarrow q-r\in E$, where $E = 2\,\mathbb{Z}$ is the set of even integers.

b) In $\mathbb{R}^2$: $(x_1,y_1)\sim(x_2,y_2) \Leftrightarrow x_1^3 + y_1 = x_2^3 + y_2$.

c) In a group $(G,\cdot)$: $a\sim b \Leftrightarrow$there is an integer k such that $a^k = b^k$.