Math 31: Abstract Algebra
Fall 2016 - Homework 6

Return date: Wednesday 10/26/16

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keywords: *quotient groups, generating sets*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 1. *(6 points)* Let $(\mathbb{R}^2, +)$ be the group $\mathbb{R}^2$, where $+$ denotes the vector addition. Let

$$N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, y = 2x \right\}.$$

a) Show that N is a subgroup. Why is N normal?

b) In geometrical terms, describe N and the elements of $\mathbb{R}^2 / N$.

c) In geometrical terms or otherwise, describe the operation of $\mathbb{R}^2 / N$.

exercise 2. *(3 points)* Let $(G, \cdot)$ be a group and $N < G$ be a normal subgroup and G/N be the corresponding quotient. Show that for $a \in G, Na \in G/N$: $\text{ord}(Na)$ is a factor of $\text{ord}(a)$.

exercise 3. *(3 points)* Let $(G, \cdot)$ be a group and $N < G$ be a normal subgroup and G/N be the corresponding quotient. Show that:

a) If G is abelian, then G/N is abelian.

b) If G is cyclic, then G/N is cyclic.

c) Find a normal subgroup N of $(S_3, \circ)$, such that S_3/N is abelian and cyclic.

Note: The example in c) shows that the inverse direction in a) and b) is not true.

exercise 4. *(3 points)* Consider $(\mathbb{Z}_n, +_n)$. If m is a factor of n, then by **Ch.11, B4**, there is a cyclic subgroup $\langle k \rangle$ of order m in $(\mathbb{Z}_n, +_n)$. Show that the quotient group $\mathbb{Z}_n / \langle k \rangle$ is a cyclic group of order $\frac{n}{m}$.
Give a counterexample that shows that in general $\mathbb{Z}_n / \langle k \rangle \times \langle k \rangle$ is not isomorphic to $(\mathbb{Z}_n, +_n)$.

exercise 5. *(5 points)* Let $(G, \cdot)$ and $(H, \cdot)$ be two groups, where S is a subset of G, such that $G = \langle S \rangle$ and T is a subset of H, such that $H = \langle T \rangle$.
Show that $(S, e_H) \cup (e_G, T)$, where

$$(S, e_H) = \{(s, e_H) : s \in S\} \quad \text{and} \quad (e_G, T) = \{(e_G, t) : t \in T\}$$

is a generating set for the group $G \times H$, i.e. $\langle (S, e_H) \cup (e_G, T) \rangle = G \times H$.
