Math 31: Abstract Algebra
Fall 2016 - Homework 3

Return date: Wednesday 10/05/16

keyword: *isomorphisms, Cayley graphs, equivalence relations*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 1. *(5 points)* Consider the groups $(\mathbb{Z}_4, +_4)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +_2 \times +_2)$ with four elements.

a) Draw up the operation table for $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

b) Draw the Cayley graphs $\Gamma(\mathbb{Z}_4, \{1\})$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(0,1),(1,0)\})$.

c) Show that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic.

exercise 2. *(6 points)* Let $\{e, a, b, c\}$ be a set of four elements, where e denotes the neutral element. Find all possible operation tables that are the operation tables of groups. Then show that in each case the group is either isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.
Note: It is sufficient to write down the corresponding group isomorphism $f : G_1 \to G_2$, where G_2 is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The condition $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in G_1$ does not have to be checked.

exercise 3. *(3 points)* Let $(G, \cdot)$ be a group with neutral element e. Let S be a generating set, i.e. $G = \langle S \rangle$ consisting of n elements. Let $\Gamma(G, S)$ be the corresponding Cayley graph. Show that

$$\mathrm{val}(h) = \mathrm{val}(e) = 2n \quad \text{for all} \quad h \in G.$$

exercise 4. *(6 points)* Prove that each of the following is an equivalence relation on the indicated set. Then describe the partition associated with the equivalence relation.

a) In $\mathbb{Q}$: $q \sim r \Leftrightarrow q - r \in \mathbb{Z}$.

b) In $\mathbb{R}^2$: $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$.

c) In a group $(G, \cdot)$: $a \sim b \Leftrightarrow$ there is an $x \in G$, such that $a = xbx^{-1}$.