Name: ______________________________

Math 31 In-class Final

December 8, 2009

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

Honor Statement:

I have neither given nor received help on this exam, and all of the answers are my own.

____

Signature

Question	Points	Score
1	18	
2	24	
3	30	
4	38	
5	30	
Total:	140	

1. Mark each of the following statements TRUE or FALSE. You do not need to justify your answers.

   (a) [2 points] Let G be a finite group and $H \leq G$. Then the index of H in G is $|G|/|H|$.

   (b) [2 points] For a ring homomorphism $\varphi : R \to S$, the kernel is defined as $\ker \varphi = \{r \in R \mid \varphi(r) = 1\}$.

   (c) [2 points] If R is a principal ideal domain, so is $R[x]$.

   (d) [2 points] In an integral domain, maximal ideals are prime ideals.

   (e) [2 points] If K is a finite extension of a field F and $a \in K$, then a is algebraic over F.

   (f) [2 points] A subset I of a ring R is an ideal if $ab \in I$ implies $a \in I$ or $b \in I$.

   (g) [2 points] If I is an ideal of a ring R, then R/I is an integral domain if and only if R is an integral domain.

   (h) [2 points] Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$. If there is a prime $p \in \mathbb{Z}$ with $p \not| a_n$, $p | a_{n-1}, \ldots p | a_0$, and $p^2 \not| a_0$, then $f(x)$ is reducible over $\mathbb{Q}$

   (i) [2 points] Let G be a group and $a \in G$ be a fixed element. Then the center of G is a normal subgroup of the centralizer of a ($Z(G) \trianglelefteq C(a)$).

2. Definitions & Examples: Give a complete definition and an example of each of the following terms. You do not need to justify your examples.

   (a) [6 points] A dihedral group.

   (b) [6 points] A maximal ideal. (Here, you'll need to provide both the ideal and the ring containing it.)

(c) [6 points] A zero divisor. (Here, you'll need to provide the zero divisor and the ring containing it.)

(d) [6 points] Finite field extension. (Here, you'll need to provide the base field as well as the extension field.)

3. Examples & Non-examples.

   (a) [8 points] Give an example of an integral domain which is not a unique factorization domain. Demonstrate that this ring is not a UFD by providing an example of an element which does not factor uniquely. [1]

   (b) [8 points] Give an example of a unique factorization domain which is not a principal ideal domain. Demonstrate that this ring is not a PID by providing an example of an ideal which is not principally generated. [2]

[1] You do NOT need to show that your ring is an integral domain, and you do NOT need to show that the factors of your example element are irreducible, although they should be.

[2] You do NOT need to show that your ring is a UFD, and you do NOT need to show that your example ideal is not principally generated, although it shouldn't be.

(c) [8 points] Give an example of a ring with unity which is not a commutative ring. Provide an example illustrating that this ring is not commutative. [3]

(d) [6 points] Give an example of a finite group which is not cyclic. You do not need to show that it is not cyclic.

[3] You do NOT need to show that your ring is a ring with unity.

4. Short Answer.

   (a) [7 points] Let K be some extension field of $\mathbb{Q}$ and suppose $\alpha \in K$ is a root of $f(x) = x^4 - 1$. Could $f(x) = x^4 - 1$ be the minimal polynomial for α over $\mathbb{Q}$? Why or why not?

(b) [10 points] Let $f(x) = x^3 + 4x^2 - 14x + 5$ and $g(x) = x - 2$ in $\mathbb{Q}[x]$. Find the quotient $q(x)$ and remainder $r(x)$ (both in $\mathbb{Q}[x]$) obtained by dividing $f(x)$ by $g(x)$.

(c) [21 points] Let $f(x) = 2x^2 + 6$. Is $f(x)$ reducible over $\mathbb{Z}$? Over $\mathbb{Q}$? Over $\mathbb{C}$? If so, give the factorization. If not, explain.

5. (Short) Proof.

   (a) [10 points] Show that if p is a prime element of an integral domain, then p is irreducible.

(b) [10 points] Let R be a commutative ring with unity. Prove that if $a \in R$ is a unit, then a has a unique inverse in R.

(c) [10 points] Let G be a group. Show that the center of G, $Z(G) = \{g \in G \mid gx = xg \ \forall x \in G\}$, is a subgroup of G.