Instructions: You are encouraged to work out solutions to these problems in groups! Discuss the problems with your classmates and/or your instructor. After doing so, please write up your solutions legibly on a separate sheet (or sheets) of paper (this part should be done on your own). Be sure to use *complete sentences*. Note: Proofs should contain *words*, not just symbols.

1. Define the *center* of a group G, $Z(G)$, to be the subset of elements in G that commute with every element of G. That is, $Z(G) = \{g \in G \mid gh = hg \ \forall\ h \in G\}$. Define the *centralizer* of an element $h \in G$, $C(h)$ or $C_G(h)$, to be the set of elements in G which commute with h. That is, $C(h) = \{g \in G \mid gh = hg\}$.

   (a) Let Q be the group of quaternions (the group you calculated the Cayley table for in problem 1 of last week's homework). What is $Z(Q)$?

   (b) Find the centralizer of each element in Q.

   (c) Notice that for each $q \in Q$, $\langle q \rangle \leq C(q)$. Is this always true? If yes, explain. If no, give a counterexample.

2. (Chapter 3, Exercise 4) Prove that in any group, an element and its inverse have the same order. What does this mean for $\langle a \rangle$ and $\langle a^{-1} \rangle$? Prove it.

3. (Chapter 3, Exercise 18) If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G.

4. (Chapter 3, Exercise 34) Suppose a and b are group elements such that $|a| = 2$, $b \neq e$, and $aba = b^2$. Determine $|b|$.

5. (Chapter 4, Exercise 14) Suppose that a cyclic group G has exactly three subgroups: G itself, $\{e\}$, and a subgroup of order 7. What is $|G|$? What can you say if 7 is replaced with p where p is a prime?

6. (Chapter 4, Exercise 32) Determine the subgroup lattice for Z_{12}. *Note:* In class we discussed how to find all the subgroups of a cyclic group. A subgroup lattice is a diagram which depicts the containment relationships between these. See pages 80-81 in your textbook for an example.

7. (Chapter 4, Exercise 40) Let m and n be elements of the group $\mathbb{Z}$. Find a generator for the group $\langle m \rangle \cap \langle n \rangle$.

8. (Chapter 4, Exercise 44) Suppose that G is a cyclic group and that 6 divides $|G|$. How many elements of order 6 does G have? If 8 divides $|G|$, how many elements of order 8 does G have? If a is one element of order 8, list the other elements of order 8.

9. (Chapter 5, Exercise 18) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$. Write α, β, and $\alpha\beta$ as

   (a) products of disjoint cycles,

   (b) products of 2-cycles.