September 30, 2008 Names:

Lab #1

Center and Centralizer

For this lab you may use your text book and class notes. Your group should work together to solve the problems and turn in one paper for your entire group.

1. Find the definitions of the *center of a group* and the *centralizer of a group element.* Write down formal definitions as well as intuitive definitions.

- $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}$
- The center of a group is the set of elements that commute with everything in the group.
- $C(a) = \{g \in G \mid ag = ga\}$
- The centralizer of an element is the set of all group elements that commute with that element.

2. Remember your first favorite group, D_4. Compute the center of D_4 and the centralizer of each element. (The Cayley table on page 33 may be helpful.)

$Z(D_4) = \{R_0, R_{180}\}$

$C(R_0) = C(R_{180}) = D_4$

$C(R_{90}) = C(R_{270}) = \{R_0, R_{90}, R_{180}, R_{270}\}$

$C(V) = C(H) = \{R_0, R_{180}, V, H\}$

$C(D) = C(D') = \{R_0, R_{180}, D, D'\}$

3. List some interesting things you notice. Are there relationships between the center and various centralizers? Is one always contained in the other? Is every element of G always in some centralizer.

$Z(D_4) \subseteq C(a)$ for any a

If $g \in Z(G)$ then $C(g) = G$.

If $g \notin Z(G)$ with $g' \in C(g)$, then $C(g) = C(g')$.

4. Carefully read through the proof that $Z(G) \leqslant G$ on page 65. Now prove for each a in a group G, the centralizer of a is a subgroup of G.

Two Step Subgroup Test

Suppose that $x, y \in C(a)$. Then $xa = ax$ & $ya = ay$.

WTS (1) $xy \in C(a)$

(2) $x^{-1} \in C(a)$

(1) $(xy)a = x(ya) \underset{\substack{\uparrow \\ y \in C(a)}}{=} x(ay) = (xa)y \underset{\substack{\uparrow \\ x \in C(a)}}{=} (ax)y = a(xy)$

So $xy \in C(a)$. ✓

(2)
$$
\begin{aligned}
xa &= ax \\
x^{-1}(xa) &= x^{-1}(ax) \\
(x^{-1}x)a &= (x^{-1}a)x \\
ea &= (x^{-1}a)x \\
a &= (x^{-1}a)x \\
ax^{-1} &= (x^{-1}a)(xx^{-1}) \\
ax^{-1} &= (x^{-1}a)e
\end{aligned}
$$

$\Rightarrow ax^{-1} = x^{-1}a$

So $x^{-1} \in C(a)$ ✓

Thus by the 2-Step Subgroup Test $C(a) \leq G$.

5. Let G be a group, and let $a \in G$. Prove that $C(a) = C(a^{-1})$.

We must show that $C(a) \subseteq C(a^{-1})$ & $C(a^{-1}) \subseteq C(a)$

$(\subseteq)$ Let $x \in C(a)$. Then $xa = ax \Rightarrow a^{-1}(xa)a^{-1} = a^{-1}(ax)a^{-1}$

$\Rightarrow (a^{-1}x)(aa^{-1}) = (a^{-1}a)(xa^{-1}) \Rightarrow (a^{-1}x)e = e(xa^{-1})$

$\Rightarrow a^{-1}x = xa^{-1}$, so $x \in C(a^{-1})$. ✓

$(\supseteq)$ We already know that $C(a) \subseteq C(a^{-1})$ for any $a \in G$, including a^{-1}. So

$C(a^{-1}) \subseteq C((a^{-1})^{-1}) = C(a)$ ✓

6. Let G be a group and consider the set $\bigcap_{a \in G} C(a)$. Use your calculations in Problem 2, to find $\bigcap_{a \in D_4} C(a)$. Is this what you expected? Use this information to conjecture a theorem. Prove your theorem.

$$\bigcap_{a \in D_4} C(a) = \underset{a = R_0,\ R_{180}}{D_4} \cap \underset{a = R_{90},\ R_{270}}{\{R_0, R_{90}, R_{180}, R_{270}\}} \cap \underset{a = V, H}{\{R_0, R_{180}, V, H\}}$$

$$\cap \underset{a = D, D'}{\{R_0, R_{180}, D, D'\}}$$

$$= \{R_0, R_{180}\} = Z(D_4)$$

Thm: For a group G, $\bigcap_{a \in G} C(a) = Z(G)$.

Proof/ $(\subseteq)$ Let $x \in \bigcap_{a \in G} C(a)$. Then $x \in C(a)$ for all $a \in G$

So $xa = ax$ for all $a \in G$. Thus $x \in Z(G)$. ✓

$(\supseteq)$ Let $x \in Z(G)$. So, $xa = ax$ for all $a \in G$.

For any $a \in G$ $x \in C(a)$. Thus $x \in \bigcap_{a \in G} C(a)$. ✓