# Intgration by Parts: Solutions

1. Consider the integral:

$$\int x\cos(5x)\mathrm{d}x$$

Let

$$u = x \qquad v = \frac{1}{5}\sin(5x)$$
$$du = dx \qquad dv = \cos(5x)dx$$

So

$$\begin{aligned}\int x\cos(x)\mathrm{d}x &= \frac{1}{5}x\sin(5x) - \frac{1}{5}\int \sin(5x)\mathrm{d}x \\ &= \frac{1}{5}x\sin(5x) - \frac{1}{5}\left(\frac{-1}{5}\cos(5x) + C\right) \\ &= \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x) + C\end{aligned}$$

2. Consider the integral:

$$\int xe^{-3x}\mathrm{d}x$$

Let

$$u = x \qquad v = \frac{-1}{3}e^{-3x}$$
$$du = dx \qquad dv = e^{-3x}dx$$

So

$$\begin{aligned}\int xe^{-3x}\mathrm{d}x &= \frac{-1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\mathrm{d}x \\ &= \frac{-1}{3}xe^{-3x} + \frac{1}{3}\left(\frac{-1}{3}e^{-3x} + C\right) \\ &= \frac{-1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C\end{aligned}$$

3. Consider the integral:

$$\int \cos(x)e^{2x}\mathrm{d}x$$

Let

$$u = \cos(x) \qquad v = \frac{1}{2}e^{2x}$$
$$du = -\sin(x)dx \qquad dv = e^{2x}dx$$

So

$$\int \cos(x)e^{2x}\mathrm{d}x = \frac{1}{2}\cos(x)e^{2x} + \frac{1}{2}\int \sin(x)e^{2x}\mathrm{d}x$$

We have to do integration by parts again so let

$$u^* = \sin(x) \qquad v^* = \frac{1}{2}e^{2x}$$
$$du^* = \cos(x)dx \qquad dv^* = e^{2x}dx$$

So

$$\int \sin(x)e^{2x}\mathrm{d}x = \frac{1}{2}\sin(x)e^{2x} - \frac{1}{2}\int \cos(x)e^{2x}\mathrm{d}x$$

We recognize that we have the same integral that we start with and so

$$\begin{aligned}\int \cos(x)e^{2x}\mathrm{d}x &= \frac{1}{2}\cos(x)e^{2x} + \frac{1}{2}\left(\frac{1}{2}\sin(x)e^{2x} - \frac{1}{2}\int \cos(x)e^{2x}\mathrm{d}x\right)\\ &= \frac{1}{2}\cos(x)e^{2x} + \frac{1}{4}\sin(x)e^{2x} - \frac{1}{4}\int \cos(x)e^{2x}\mathrm{d}x\end{aligned}$$

Thus we have

$$\frac{5}{4}\int \cos(x)e^{2x}\mathrm{d}x = \frac{1}{2}\cos(x)e^{2x} + \frac{1}{4}\sin(x)e^{2x} + C$$

and so

$$\int \cos(x)e^{2x}\mathrm{d}x = \frac{2}{5}\cos(x)e^{2x} + \frac{1}{5}\sin(x)e^{2x} + C$$

4. Consider the integral:

$$\int (\ln(x))^2\mathrm{d}x$$

Let

$$u = (\ln(x))^2 \qquad v = x$$
$$du = \frac{2\ln(x)}{x}dx \qquad dv = dx$$

So

$$\int (\ln(x))^2 \mathrm{d}x = x\ln(x)^2 - 2\int \ln(x)\mathrm{d}x$$

We have to do integration by parts again so let

$$\begin{aligned} u^* &= \ln(x) & v^* &= x \\ du^* &= \frac{1}{x}dx & dv^* &= dx \end{aligned}$$

So

$$\begin{aligned} \int \ln(x)\mathrm{d}x &= x\ln(x) - \int \mathrm{d}x \\ &= x\ln(x) - x + C \end{aligned}$$

So

$$\begin{aligned} \int \sin^{-1}(x)\mathrm{d}x &= x\ln(x)^2 - 2\int \ln(x)\mathrm{d}x \\ &= x\ln(x)^2 - 2\left(x\ln(x) - x + C\right) \\ &= x\ln(x)^2 - 2x\ln(x) + 2x + C \end{aligned}$$

5. Consider the integral:

$$\int t2^t \mathrm{d}t$$

Let

$$\begin{aligned} u &= t & v &= \frac{1}{\ln(2)}2^t \\ du &= dt & dv &= 2^t dt \end{aligned}$$

So

$$\begin{aligned} \int t2^t \mathrm{d}t &= \frac{1}{\ln(2)}t2^t - \frac{1}{\ln(2)}\int 2^t \mathrm{d}t \\ &= \frac{1}{\ln(2)}t2^t - \frac{1}{\ln(2)}\left(\frac{1}{\ln(2)}2^t + C\right) \\ &= \frac{1}{\ln(2)}t2^t - \frac{1}{\ln(2)^2}2^t + C \end{aligned}$$

6. Consider the integral:

$$\int \sin(2t)\cos(t)\mathrm{d}t$$

Let

$$u = \sin(2t) \qquad v = \sin(t)$$
$$du = 2\cos(2t)dt \qquad dv = \cos(t)dt$$

So

$$\int \sin(2t)\cos(t)\mathrm{d}t = \sin(t)\sin(2t) - 2\int \cos(2t)\sin(t)\mathrm{d}t$$

We have to do integration by parts again so let

$$u^* = \cos(2t) \qquad v^* = -\cos(t)$$
$$du^* = -2\sin(2t)dt \qquad dv^* = \sin(t)dt$$

So

$$\int \cos(2t)\sin(t)\mathrm{d}t = -\cos(2t)\cos(t) - 2\int \sin(2t)\cos(t)\mathrm{d}t$$

We recognize that we have the same integral that we start with and so

$$\begin{aligned}\int \sin(2t)\cos(t)\mathrm{d}t &= \sin(t)\sin(2t) - 2\left(-\cos(2t)\cos(t) - 2\int \sin(2t)\cos(t)\mathrm{d}t\right)\\ &= \sin(t)\sin(2t) + 2\cos(2t)\cos(t) + 4\int \sin(2t)\cos(t)\mathrm{d}t\end{aligned}$$

Thus we have

$$-3\int \sin(2t)\cos(t)\mathrm{d}t = \sin(t)\sin(2t) + 2\cos(2t)\cos(t) + C$$

and so

$$\int \sin(2t)\cos(t)\mathrm{d}t = \frac{-1}{3}\sin(t)\sin(2t) + \frac{-2}{3}\cos(2t)\cos(t) + C$$

7. Consider the integral:

$$\int (x^2 + 2x)\cos(x)\mathrm{d}x$$

Let

$$u = x^2 + 2x \qquad v = \sin(x)$$
$$du = 2x + 2dx \qquad dv = \cos(x)dx$$

So

$$\int (x^2 + 2x)\cos(x)\mathrm{d}x = (x^2 + 2x)\sin(x) - \int (2x + 2)\sin(x)\mathrm{d}x$$

We have to do integration by parts again so let

$$\begin{aligned} u^* &= 2x + 2 & v^* &= -\cos(x) \\ du^* &= 2dx & dv^* &= \sin(x)dx \end{aligned}$$

So

$$\begin{aligned} \int (2x + 2)\sin(x)\mathrm{d}x &= -(2x + 2)\cos(x) + 2\int \cos(x)\mathrm{d}x \\ &= -(2x + 2)\cos(x) + 2\sin(x) + C \end{aligned}$$

So

$$\begin{aligned} \int (x^2 + 2x)\cos(x)\mathrm{d}x &= (x^2 + 2x)\sin(x) - \int (2x + 2)\sin(x)\mathrm{d}x \\ &= (x^2 + 2x)\sin(x) - (-(2x + 2)\cos(x) + 2\sin(x) + C) \\ &= (x^2 + 2x)\sin(x) + (2x + 2)\cos(x) - 2\sin(x) + C \end{aligned}$$

8. Consider the integral

$$\int \arcsin(t)\mathrm{d}t$$

Let

$$\begin{aligned} u &= \arcsin(t) & v &= t \\ du &= \frac{1}{\sqrt{1 - t^2}}dt & dv &= dt \end{aligned}$$

So

$$\int \sin^{-1}(x)\mathrm{d}x = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1 - x^2}}\mathrm{d}x$$

We now do the substitution method with $w = 1 - x^2$ and $dw = -2xdx$ to get

$$\begin{aligned} \int \sin^{-1}(x)\mathrm{d}x &= x\sin^{-1}(x) - \int \frac{x}{\sqrt{1 - x^2}}\mathrm{d}x \\ &= x\sin^{-1}(x) + \frac{1}{2}\int \frac{1}{\sqrt{w}}\mathrm{d}w \\ &= x\sin^{-1}(x) + \frac{1}{2}2w^{1/2} + C \\ &= x\sin^{-1}(x) + \sqrt{1 - x^2} + C \end{aligned}$$