Midterm 1 Review Sheet

List of Topics:

- Derivative Rules and Basic Derivatives
  - Chain Rule
  - Product Rule
  - Basic Derivatives: polynomials, trigonometric functions, $ln(x)$, e^x, $x^{n/m}$
- Rectangular Approximations (Riemann Sums)
  - Left end point (L_n)
  - Right end point (R_n)
  - Midpoint (M_n)
- Definite Integral
  - $\int_a^b f(x)\,dx$
  - Definition in term of Riemann Sums
  - Geometric Interpretation
- Fundamental Theorem of Calculus part 1
  - Statement of Theorem
  - Applications
- Fundamental Theorem of Calculus part 2
  - Statement of Theorem
  - Applications
- Indefinite Integral
  - $\int f(x)\,dx$
  - Don't forgot to add "$+C$" to your anti-derivative
- u - Substitution (reverse chain rule)
  - $du = (\frac{du}{dx})dx = u'dx$
  - Recognize when to use it
  - $u =$ the "nested" function
  - $u =$ the function whose derivative is sitting outside
  - You can use either x bounds on an anti-derivative written in terms of x, or use the u bounds on an anti-derivative written in terms of u.
- Integration by Parts (reverse product rule)
  - $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$
  - Use it when either told to or when u - Substitution fails
  - Remember that the "parts" (u and dv) are parts of a product
  - You have a limited number of choices for what u and dv can be, sometimes recognizing a u - Substitution integral as part of the product is necessary to determine your choices for u and dv.

Representative sample of problems

Derivatives Practice:

Find the Derivative

(i)

$$f(x) = e^{x^2} sin(ln(x))$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(e^{x^2} sin(ln(x))\right) \\ &= 2xe^{x^2}\sin(\ln(x)) + e^{x^2}\left[\frac{1}{x}\cos(\ln(x))\right] \end{aligned}$$

(ii)

$$f(x) = ln(tan(e^{(x^2+x)}))$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(ln(tan(e^{(x^2+x)}))\right) \\ &= \frac{1}{\tan\left(e^{x^2+x}\right)}\left(\sec\left(e^{x^2+x}\right) e^{x^2+x}(2x+1)\right) \end{aligned}$$

(iii)

$$f(x) = \frac{arcsin(ln(x))}{e^x(x^4 - 3x^3 + x - e)}$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{arcsin(ln(x))}{e^x(x^4 - 3x^3 + x - e)}\right) \\ &= \frac{\left(1-(\ln(x))^2\right)^{\frac{1}{2}}(4x^4 - 3x^3 + x - e) - x\arcsin(\ln(x))\left[(x^4 - 3x^3 + x - e) + (4x^3 - 9x^2 + 1)\right]}{xe^x(x^4 - 3x^3 + x - e)^2} \end{aligned}$$

(iv)

$$f(x) = ln(ln(x))e^{sin(x)} - sin(sin(sin(x)))$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(ln(ln(x))e^{sin(x)} - sin(sin(sin(x)))\right) \\ &= \frac{1}{x\ln(x)}e^{\sin(x)} + \ln(\ln(x))\cos(x)e^{\sin(x)} - \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x) \end{aligned}$$

Rectangular Approximations (Riemann Sums):

Find L_n, R_n, and M_n for the given function on the given interval:

(Remember that n tells you how many equal sized pieces to break the interval into to use as the base of your rectangles)

(i) $f(x) = 2x^3 - x^2 + 1$ Interval: $[0, 3]$ $n = 3$

$$\begin{aligned} L_n &= \frac{3-0}{3}\left(f(0) + f(1) + f(2)\right) = 1\left(0 + 2 + 13\right) = 15 \\ R_n &= \frac{3-0}{3}\left(f(1) + f(2) + f(3)\right) = 1\left(2 + 13 + 46\right) = 61 \\ M_n &= \frac{3-0}{3}\left(f(1/2) + f(3/2) + f(5/2)\right) = 1\left(1 + 22/4 + 104/4\right) = 130/4 \end{aligned}$$

(ii) $f(x) = sin(x) + 1/2$ Interval: $[-2\pi, 2\pi]$ $n = 4$

$$\begin{aligned} L_n &= \frac{2\pi - (-2\pi)}{4}\left(f(-2\pi) + f(-\pi) + f(0) + f(\pi)\right) = \pi\left(1/2 + 1/2 + 1/2 + 1/2\right) = 2\pi \\ R_n &= \frac{2\pi - (-2\pi)}{4}\left(f(-\pi) + f(0) + f(\pi) + f(2\pi)\right) = \pi\left(1/2 + 1/2 + 1/2 + 1/2\right) = 2\pi \\ M_n &= \frac{2\pi - (-2\pi)}{4}\left(f(-3\pi/2) + f(-\pi/2) + f(\pi/2) + f(3\pi/2)\right) = \pi\left(3/2 + -1/2 + 3/2 + -1/2\right) = 2\pi \end{aligned}$$

(iii) $f(x) = 3x + 2$ Interval: $[-3, -2]$ $n = 5$

$$\begin{aligned} L_n &= \frac{-3-(-3)}{5}\left(f\left(\frac{-15}{5}\right) + f\left(\frac{-14}{5}\right) + f\left(\frac{-13}{5}\right) + f\left(\frac{-12}{5}\right) + f\left(\frac{-11}{5}\right)\right) \\ &= \frac{1}{5}\left(\frac{-35}{5} + \frac{-32}{5} + \frac{-29}{5} + \frac{-26}{5} + \frac{-23}{5}\right) = \frac{-29}{5} \\ R_n &= \frac{-3-(-3)}{5}\left(f\left(\frac{-14}{5}\right) + f\left(\frac{-13}{5}\right) + f\left(\frac{-12}{5}\right) + f\left(\frac{-11}{5}\right) + f\left(\frac{-10}{5}\right)\right) \\ &= \frac{1}{5}\left(\frac{-32}{5} + \frac{-29}{5} + \frac{-26}{5} + \frac{-23}{5} + \frac{-20}{5}\right) = \frac{-26}{5} \\ M_n &= \frac{-3-(-3)}{5}\left(f\left(\frac{-29}{10}\right) + f\left(\frac{-27}{10}\right) + f\left(\frac{-25}{10}\right) + f\left(\frac{-23}{10}\right) + f\left(\frac{-21}{10}\right)\right) \\ &= \frac{1}{5}\left(\frac{-67}{10} + \frac{-61}{10} + \frac{-55}{10} + \frac{-49}{10} + \frac{-43}{10}\right) = \frac{-11}{2} \end{aligned}$$

The definite integral:

Definition: Let f be a continuous function on the interval $[a, b]$. The definite integral of f over the interval $[a, b]$, denoted $\int_a^b f(x)\,dx$, is defined to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n$$

(all of the limits have the same value). Alternatively you may also, more generally speaking, break the interval into n equal sized pieces with the size denoted $\Delta x = \frac{b-a}{n}$ and choose any "sample point" from each piece of the partition, denoted x_i^* where $1 \leq i \leq n$ for each piece of the partition of the interval into n pieces, and define the definite integral as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*)$$

Use the geometric interpretation of the definite integral to find the following:

(i)

$$\int_1^3 4x - 2\,dx$$

$$\int_1^3 4x - 2\mathrm{d}t = 12$$

After drawing the line, we see the region can be expressed as the sum of a rectangle and a triangle. the rectangle has width 2 and height 2 for an area of 4. And the triangle has width 2 and height 8 for an area of 8. Thus the total area is 12.

(ii)

$$\int_{-3}^{3} \sqrt{9 - t^2}\,dt$$

$$\int_{-3}^{3} \sqrt{9 - t^2}\mathrm{d}t = \frac{9\pi}{2}$$

We recognize that this function is the upper half of a circle centred at the origin with radius 3. Thus the area is half of a full circle of radius three. Since the radius of the full circle would be 9π we just divide by 2.

(iii)

$$\int_{-10}^{10} sin(\theta)\,d\theta$$

$$\int_{-10}^{10} \sin(\theta)\mathrm{d}\theta = 0$$

Recall that $\sin(\theta)$ is an odd function, and so $\sin(-\theta) = -\sin(\theta)$. Thus the Riemann Sum from 0 to 10 is the negative of the Riemann Sum from -10 to 0. And so the two parts cancel out.

Fundamental Theorem of Calculus part 1:

Theorem: If f is a continuous function on $[a, b]$, then we can define a function g by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

In which case g will be continuous on $[a, b]$, differentiable on (a, b) and (most importantly)

$$g'(x) = f(x).$$

Use the Fundamental Theorem of Calculus to find the derivatives of the following functions:

(i)

$$f(x) = \int_1^x ln(u)e^{u^2}\,du$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_1^x \mathrm{ln}(u)e^{u^2}\mathrm{d}u \\ &= ln(x)e^{x^2} \end{aligned}$$

(ii)

$$f(x) = \int_x^2 sin(ln(sec(t)))\,dt$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_x^2 sin(ln(sec(t)))\mathrm{d}t \\ &= -\frac{d}{dx}\int_2^x sin(ln(sec(t)))\mathrm{d}t \\ &= -sin(ln(sec(x))) \end{aligned}$$

(iii)

$$f(x) = \int_0^{x^5} e^{t^2}\,dt$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_0^{x^5} e^{t^2}\mathrm{d}t \\ &= e^{\left(x^5\right)^2}5x^4 \\ &= 5e^{\left(x^{10}\right)}x^4 \end{aligned}$$

(iv)

$$f(x) = \int_{x^2}^{sin(x)} e^{t^7}\,dt$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_{x^2}^{\sin(x)} e^{t^7}\mathrm{d}t \\ &= \frac{d}{dx}\left[\int_0^{\sin(x)} e^{t^7}\mathrm{d}t + \int_{x^2}^0 e^{t^7}\mathrm{d}t\right] \\ &= \frac{d}{dx}\left(\int_0^{\sin(x)} e^{t^7}\mathrm{d}t\right) - \frac{d}{dx}\left(\int_0^{x^2} e^{t^7}\mathrm{d}t\right) \\ &= e^{\sin(x)^7}\cos(x) - 2e^{x^{14}}x \end{aligned}$$

Fundamental Theorem of Calculus part 2:

Theorem: If f is continuous on $[a, b]$ and F is any anti-derivative of f (i.e. $F' = f$) then

$$\int_a^b f(x)\, dx = F(a) - F(b)$$

Evaluate the following definite integrals:

(i)

$$\int_0^{\pi} sin(x)\, dx$$

$$\begin{aligned}\int_0^{\pi} \sin(x)\mathrm{d}x &= -\cos(x)\Big|_0^{\pi} \\ &= -\cos(\pi) + \cos(0) \\ &= -(-1) + 1 \\ &= 2\end{aligned}$$

(ii)

$$\int_0^1 e^u\, du$$

$$\begin{aligned}\int_0^1 e^u \mathrm{d}u &= e^u\Big|_0^1 \\ &= e^1 - e^0 \\ &= e - 1\end{aligned}$$

(iii)

$$\int_0^2 \left(t^4 + 3t^2 + 5t + 2\right)\, dt$$

$$\begin{aligned}\int_0^2 \left(t^4 + 3t^2 + 5t + 2\right)\mathrm{d}t &= \left(\frac{1}{5}t^5 + t^3 + \frac{5}{2}t^2 + 2t\right)\Big|_0^2 \\ &= \frac{1}{5}2^5 + 2^3 + \frac{5}{2}2^2 + 2(2) \\ &= \frac{117}{5}\end{aligned}$$

(iv)

$$\int_{-\pi}^{-\pi/2} \left(-cos(x) + \frac{1}{3x}\right)\, dx$$

$$\begin{aligned}\int_{-\pi}^{\frac{-\pi}{2}} \left(-cos(x) + \frac{1}{3x}\right)\mathrm{d}x &= \left(-\sin(x) + \frac{1}{3}\ln(|x|)\right)\Big|_{-\pi}^{\frac{-\pi}{2}} \\ &= \left(-\sin(\frac{-\pi}{2}) + \frac{1}{3}\ln\left(\frac{\pi}{2}\right)\right) - \left(-\sin(\pi) + \frac{1}{3}\ln(\pi)\right) \\ &= 1 + \frac{1}{3}\left(\ln\left(\frac{\pi}{2}\right) - \ln(\pi)\right) \\ &= 1 + \frac{1}{3}\ln\left(\frac{1}{2}\right)\end{aligned}$$

(v)

$$\int_1^3 u^{-11/8}\,du$$

$$\begin{aligned}\int_1^3 u^{-11/8}\mathrm{d}u &= -\frac{8}{3}u^{-\frac{3}{8}}\Big|_1^3 \\ &= -\frac{8}{3}\left[3^{-\frac{3}{8}} - 1\right]\end{aligned}$$

(vi)

$$\int_{\pi/6}^{\pi/3} (sec(\theta))^2 - 2csc(2\theta)cot(2\theta)\,d\theta$$

$$\begin{aligned}\int_{\pi/6}^{\pi/3} (sec(\theta))^2 - 2csc(2\theta)cot(2\theta)\mathrm{d}\theta &= (\tan(\theta) + \csc(2\theta))\Big|_{\pi/6}^{\pi/3} \\ &= \tan(\pi/3) - \tan(\pi/6) + \csc(2\pi/3) - \csc(\pi/3) \\ &= \sqrt{3} - \frac{1}{\sqrt{3}} \\ &= \frac{2}{\sqrt{3}}\end{aligned}$$

Indefinite Integral:

Find expressions for the following indefinite integrals (don't forgot the "$+C$"):

(i)

$$\int \frac{2}{u}\ du$$

$$\int \frac{2}{u}\mathrm{d}u = 2\ln(|u|) + C$$

(ii)

$$\int \frac{7^x}{3}\ dx$$

$$\int \frac{7^x}{3}\mathrm{d}x = \frac{7^x}{3\ln(7)} + C$$

(iii)

$$\int \left(tan(3t+1) + t^{-10/11}\right)\ dt$$

$$\int \tan(3t+1) + t^{-10/11}\mathrm{d}t = \frac{1}{3}\ln(|\sec(3t+1)|) + 11t^{1/11} + C$$

u - Substitution:

Evaluate the following integrals (definite and indefinite):

(i)

$$\int 7x^{-6}cos(x^{-5})\ dx$$

Let

$$u = x^{-5} \qquad du = -5x^{-6}dx$$

Then

$$\begin{aligned}\int 7x^{-6}cos(x^{-5})\mathrm{d}x &= -\frac{7}{5}\int cos(u)\mathrm{d}u \\ &= -\frac{7}{5}\sin(u) + C \\ &= -\frac{7}{5}\sin(x^{-5}) + C\end{aligned}$$

(ii)

$$\int \frac{1}{3}x^5e^{2x^6}\ dx$$

Let

$$u = 2x^6 \qquad du = 12x^5dx$$

Then

$$\begin{aligned}\int \frac{1}{3}x^5e^{2x^6}\mathrm{d}x &= \frac{1}{36}\int e^u\mathrm{d}u \\ &= \frac{1}{36}e^u + C \\ &= \frac{1}{36}e^{2x^6} + C\end{aligned}$$

(iii)

$$\int (4x^2+1)(4x^3+3x)^{2/3}\ dx$$

Let

$$u = 4x^3 + 3x \qquad du = (12x^2+3)dx$$

Then

$$\begin{aligned}\int (4x^2+1)(4x^3+3x)^{2/3}\mathrm{d}x &= \frac{1}{3}\int u^{2/3}\mathrm{d}u \\ &= \frac{1}{5}u^{5/3} + C \\ &= \frac{1}{5}(4x^3+3x)^{5/3} + C\end{aligned}$$

(iv)

$$\int_0^{\pi/4} (sin(t))^3cos(t)\ dt$$

Let

$$u = \sin(t) \qquad du = \cos(t)dt$$

Then

$$\begin{aligned}\int_0^{\pi/4} (sin(t))^3 cos(t)\mathrm{d}t &= \int_{t=0}^{t=\pi/4} u^3 \mathrm{d}u \\ &= \frac{1}{4}u^4\Big|_{t=0}^{t=\pi/4} \\ &= \frac{1}{4}\sin(t)^4\Big|_0^{\pi/4} \\ &= \frac{1}{4}\left(sin\left(\frac{\pi}{4}\right)^4 - \sin(0)^4\right) \\ &= \frac{1}{16}\end{aligned}$$

(v)

$$\int_{1/2}^{1} \frac{ln(2t)}{3t}\ dt$$

Let

$$u = \ln(2t) \qquad du = \frac{1}{2t}dt$$

Then

$$\begin{aligned}\int_{1/2}^{1} \frac{ln(2t)}{3t}\mathrm{d}t &= \frac{2}{3}\int_{t=1/2}^{t=1} u\mathrm{d}u \\ &= \frac{1}{3}u^2\Big|_{t=1/2}^{t=1} \\ &= \frac{1}{3}\ln(2t)^2\Big|_{1/2}^{1} \\ &= \frac{1}{3}\left(\ln(2)^2 - 0\right) \\ &= \frac{1}{3}\ln(2)^2\end{aligned}$$

(vi)

$$\int_0^{\pi/4} sec^2(\theta)tan(\theta)\ d\theta$$

Let

$$u = \tan(\theta) \qquad du = \sec(\theta)^2 d\theta$$

Then

$$\begin{aligned}\int_0^{\pi/4} \sec^2(\theta)\tan(\theta)\mathrm{d}\theta &= \int_{\theta=0}^{\theta=\pi/4} u\mathrm{d}u \\ &= \frac{1}{2}u^2\Big|_{\theta=0}^{\theta=\pi/4} \\ &= \frac{1}{2}\tan(\theta)^2\Big|_0^{\pi/4} \\ &= \frac{1}{2}\left(\tan(\pi/4)^2 - \tan(0)^2\right) \\ &= \frac{1}{2}\end{aligned}$$

<u>Integration by Parts:</u>

Evaluate the following integrals (definite and indefinite):

(i)

$$\int 3^x 8^x \ dx$$

Let

$$u = 3^x \qquad v = \frac{1}{\ln(8)} 8^x$$

$$du = \ln(3) 3^x dx \qquad dv = 8^x dx$$

So

$$\int 3^x 8^x \mathrm{d}x = \frac{1}{\ln(8)} 8^x 3^x - \frac{\ln(3)}{\ln(8)} \int 3^x 8^x \mathrm{d}x$$

We see we got the same integral we started with and so, we move it to the other side to isolate it. Thus

$$\left(1 + \frac{\ln(3)}{\ln(8)}\right) \int 3^x 8^x \mathrm{d}x = \frac{1}{\ln(8)} 8^x 3^x + C$$

And so

$$\int 3^x 8^x \mathrm{d}x = \frac{8^x 3^x}{\ln(8) + \ln(3)} + C$$

(ii)

$$\int tln(t) \ dt$$

Let

$$u = \ln(t) \qquad v = \frac{1}{2} t^2$$

$$du = \frac{1}{t} dt \qquad dv = t dt$$

So

$$\begin{aligned} \int t \ln(t) \mathrm{d}t &= \frac{1}{2} t^2 \ln(t) - \frac{1}{2} \int t \mathrm{d}t \\ &= \frac{1}{2} t^2 \ln(t) = \frac{1}{4} t^2 + C \end{aligned}$$

(iii)

$$\int arcsin(x) \ dx$$

Let

$$u = \arcsin(x) \qquad v = x$$

$$du = \frac{1}{\sqrt{1 - x^2}} dx \qquad dv = dx$$

So

$$\int \arcsin(x) \mathrm{d}x = x \arcsin(x) - \int \frac{x}{\sqrt{1 - x^2}} \mathrm{d}x$$

We now do the substitution method with $w = 1 - x^2$ and $dw = -2xdx$ to get

$$\begin{aligned}\int \arcsin(x)\mathrm{d}x &= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x \\ &= x\arcsin(x) + \frac{1}{2}\int \frac{1}{\sqrt{w}}\mathrm{d}w \\ &= x\arcsin(x) + \frac{1}{2}2w^{1/2} + C \\ &= x\arcsin(x) + \sqrt{1-x^2} + C\end{aligned}$$

(iv)

$$\int arctan(x)\ dx$$

Let

$$\begin{aligned} u &= \arctan(x) & v &= x \\ du &= \frac{1}{1+x^2}dx & dv &= dx\end{aligned}$$

So

$$\int \arctan(x)\mathrm{d}x = x\arctan(x) - \int \frac{x}{1+x^2}\mathrm{d}x$$

We now do the substitution method with $w = 1 + x^2$ and $dw = 2xdx$ to get

$$\begin{aligned}\int \arcsin(x)\mathrm{d}x &= x\arctan(x) - \int \frac{x}{1+x^2}\mathrm{d}x \\ &= x\arctan(x) - \frac{1}{2}\int \frac{1}{w}\mathrm{d}w \\ &= x\arctan(x) - \frac{1}{2}\ln(|w|) + C \\ &= x\arctan(x) + \ln(\left|1+x^2\right|) + C\end{aligned}$$

(v)

$$\int_0^1 t^{13}e^{4t^7}\ dt$$

We first do the substitution method with $u = 4t^7$ and $du = 28t^6dt$ So

$$\int_0^1 t^{13}e^{4t^7}\mathrm{d}t = \frac{1}{112}\int_{t=0}^{t=1} ue^u\mathrm{d}u$$

We will also change the bounds to u to get

$$\begin{aligned}\int_0^1 t^{13}e^{4t^7}\mathrm{d}t &= \frac{1}{112}\int_{t=0}^{t=1} ue^u\mathrm{d}u \\ &= \frac{1}{112}\int_{u=0}^{u=4} ue^u\mathrm{d}u\end{aligned}$$

We now do integration by parts with

$$\begin{aligned} v &= u & w &= e^u \\ dv &= du & dw &= e^u du\end{aligned}$$

So

$$\begin{aligned}\frac{1}{112}\int_{u=0}^{u=4} ue^u\mathrm{d}u &= \frac{1}{112}\left[ue^u\Big|_0^4 - \int_0^4 e^u\mathrm{d}u\right] \\ &= \frac{1}{28}e^4 - \frac{1}{112}e^4 + \frac{1}{112}\end{aligned}$$

(vi)

$$\int_0^{\sqrt{\pi/2}} 3t^3 sin(t^2)\ dt$$

We first do the substitution method with $u = t^2$ and $du = 2tdt$ So

$$\int_0^{\sqrt{\pi/2}} 3t^3 sin(t^2)\mathrm{d}t = \frac{3}{2}\int_{t=0}^{t=\sqrt{\pi/2}} u\sin(u)\mathrm{d}u$$

We will also change the bounds to u to get

$$\begin{aligned}\int_0^{\sqrt{\pi/2}} 3t^3 sin(t^2)\mathrm{d}t &= \frac{3}{2}\int_{t=0}^{t=\sqrt{\pi/2}} u\sin(u)\mathrm{d}u \\ &= \frac{3}{2}\int_{u=0}^{u=\pi/2} u\sin(u)\mathrm{d}u\end{aligned}$$

We now do integration by parts with

$$\begin{aligned} v &= u & w &= -\cos(u) \\ dv &= du & dw &= \sin(u)du \end{aligned}$$

So

$$\begin{aligned}\frac{3}{2}\int_{u=0}^{u=\pi/2} u\sin(u)\mathrm{d}u &= \frac{3}{2}\left[-u\cos(u)\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos(u)\mathrm{d}u\right] \\ &= \frac{3}{2}\end{aligned}$$

(vii)

$$\int_{\pi/4}^{5\pi/4} e^{2x} sin(x)\ dx$$

Let

$$\begin{aligned} u &= \sin(x) & v &= \frac{1}{2}e^{2x} \\ du &= \cos(x)dx & dv &= e^{2x}dx \end{aligned}$$

So

$$\int_{\pi/4}^{5\pi/4} \sin(x)e^{2x}\mathrm{d}x = \frac{1}{2}\sin(x)e^{2x}\Big|_{\pi/4}^{5\pi/4} - \frac{1}{2}\int_{\pi/4}^{5\pi/4}\cos(x)e^{2x}\mathrm{d}x$$

We have to do integration by parts again so let

$$\begin{aligned} u^* &= \cos(x) & v^* &= \frac{1}{2}e^{2x} \\ du^* &= -\sin(x)dx & dv^* &= e^{2x}dx \end{aligned}$$

So

$$\int_{\pi/4}^{5\pi/4} \cos(x)e^{2x}\mathrm{d}x = \frac{1}{2}\cos(x)e^{2x}\Big|_{\pi/4}^{5\pi/4} + \frac{1}{2}\int_{\pi/4}^{5\pi/4}\sin(x)e^{2x}\mathrm{d}x$$

We recognize that we have the same integral that we start with and so

$$\begin{aligned}
\int_{\pi/4}^{5\pi/4} \sin(x)e^{2x}\mathrm{d}x &= \frac{1}{2}\sin(x)e^{2x}\Big|_{\pi/4}^{5\pi/4} - \frac{1}{2}\left(\frac{1}{2}\cos(x)e^{2x}\Big|_{\pi/4}^{5\pi/4} + \frac{1}{2}\int_{\pi/4}^{5\pi/4} \sin(x)e^{2x}\mathrm{d}x\right) \\
&= \left(\frac{1}{2}\sin(x)e^{2x} - \frac{1}{4}\cos(x)e^{2x}\right)\Big|_{\pi/4}^{5\pi/4} - \frac{1}{4}\int_{\pi/4}^{5\pi/4} \sin(x)e^{2x}\mathrm{d}x
\end{aligned}$$

Thus we have

$$\frac{5}{4}\int_{\pi/4}^{5\pi/4} \sin(x)e^{2x}\mathrm{d}x = \left(\frac{1}{2}\sin(x)e^{2x} - \frac{1}{4}\cos(x)e^{2x}\right)\Big|_{\pi/4}^{5\pi/4}$$

and so

$$\begin{aligned}
\int_{\pi/4}^{5\pi/4} \sin(x)e^{2x}\mathrm{d}x &= \frac{4}{5}\left(\frac{1}{2}\sin(x)e^{2x} - \frac{1}{4}\cos(x)e^{2x}\right)\Big|_{\pi/4}^{5\pi/4} \\
&= \frac{2}{5}\left(\sin(x)e^{2x} - \frac{1}{2}\cos(x)e^{2x}\right)\Big|_{\pi/4}^{5\pi/4} \\
&= \frac{2}{5}\left[\left(\sin(5\pi/4)e^{5\pi/2} - \frac{1}{2}\cos(5\pi/4)e^{5\pi/2}\right) - \left(\sin(\pi/4)e^{\pi/2} - \frac{1}{2}\cos(\pi/4)e^{\pi/2}\right)\right]
\end{aligned}$$