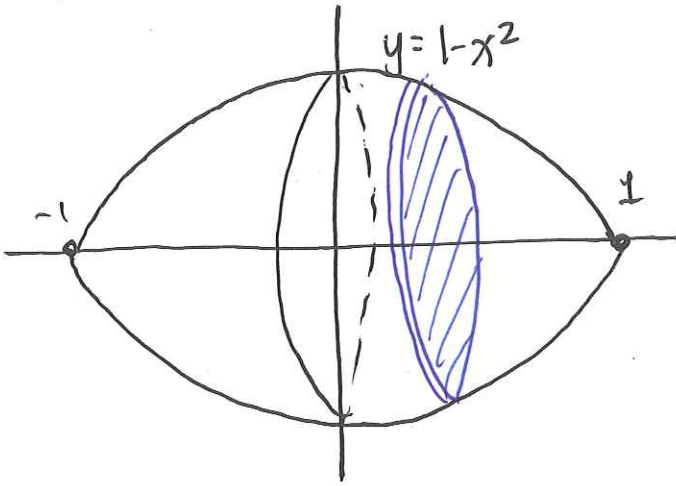
HW4 Key

6.2: 2, 4, 14, 40, 49, 61

6.3: 2, 8, 14, 38, 45, 48

6.2

2. $y = 1 - x^2$; $y = 0$; about the x-axis.



$$\text{Vol} = \int_{-1}^{1} \pi (1-x^2)^2\,dx$$

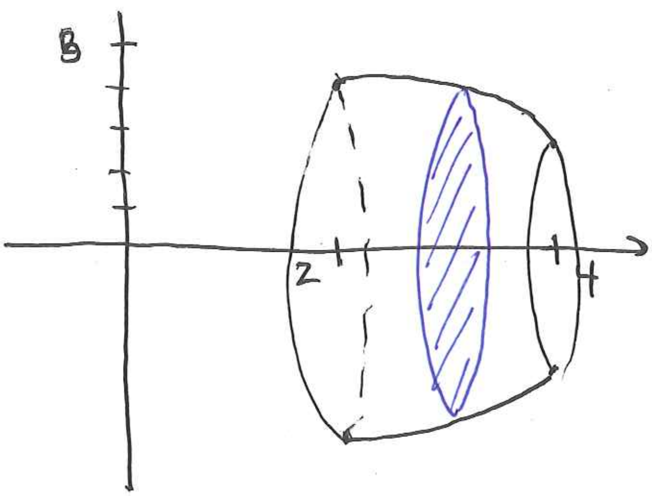
$$= \int_{-1}^{1} \pi (1 - 2x^2 + x^4)\,dx$$

$$= \pi \left(x - \frac{2}{3}x^3 + \frac{x^5}{5} \right) \Big|_{-1}^{1}$$

$$= \pi \left(1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} \right)$$

$$= \boxed{16\pi/15}$$

4. $y = \sqrt{25 - x^2}$, $y = 0$, $x = 2$, $x = 4$; about the x-axis



$$\int_2^4 \pi \left(\sqrt{25 - x^2} \right)^2 dx$$

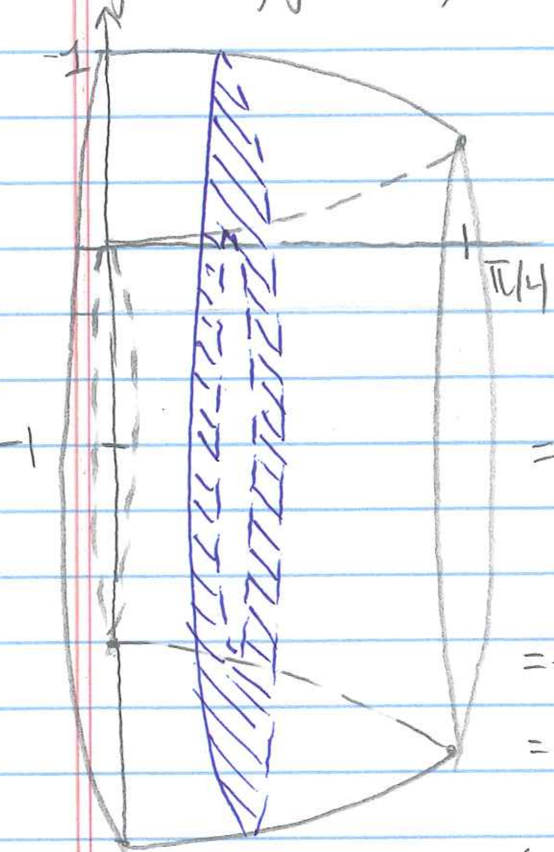
$$= \int_2^4 \pi (25 - x^2)\,dx$$

$$= \pi \left(25x - \frac{x^3}{3} \right) \Big|_2^4$$

$$= \pi \left(100 - \frac{64}{3} - 50 + \frac{8}{3} \right)$$

$$= \boxed{\frac{94\pi}{3}}$$

14. $y=\sin x, y=\cos x, 0\le x\le \pi/4$; about $y=-1$



Outer radius: $1+\cos x$
Inner radius: $1+\sin x$

Volume $= \int_0^{\pi/4} \pi(1+\cos x)^2 - \pi(1+\sin x)^2\,dx$

$= \pi\int_0^{\pi/4} (1+2\cos x+\cos^2 x-1-2\sin x-\sin^2 x)\,dx$

$= \pi\int_0^{\pi/4} 2\cos x-2\sin x+\cos^2 x-\sin^2 x\,dx$

$= \pi\int_0^{\pi/4} 2\cos x-2\sin x+\cos 2x\,dx$

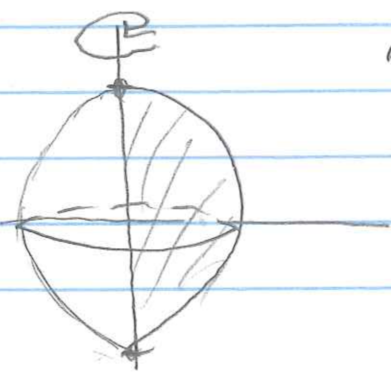
$= \pi\left(2\sin x+2\cos x+\frac{\sin 2x}{2}\right)\Big|_0^{\pi/4}$

$= \pi\left(2\left(\frac{\sqrt{2}}{2}\right)+2\left(\frac{\sqrt{2}}{2}\right)+\frac{\sin \pi/2}{2}-2\right)$

$= \boxed{\pi\left(2\sqrt{2}-3/2\right)}$

40. $\pi\int_{-1}^{1}(1-y^2)^2\,dy$

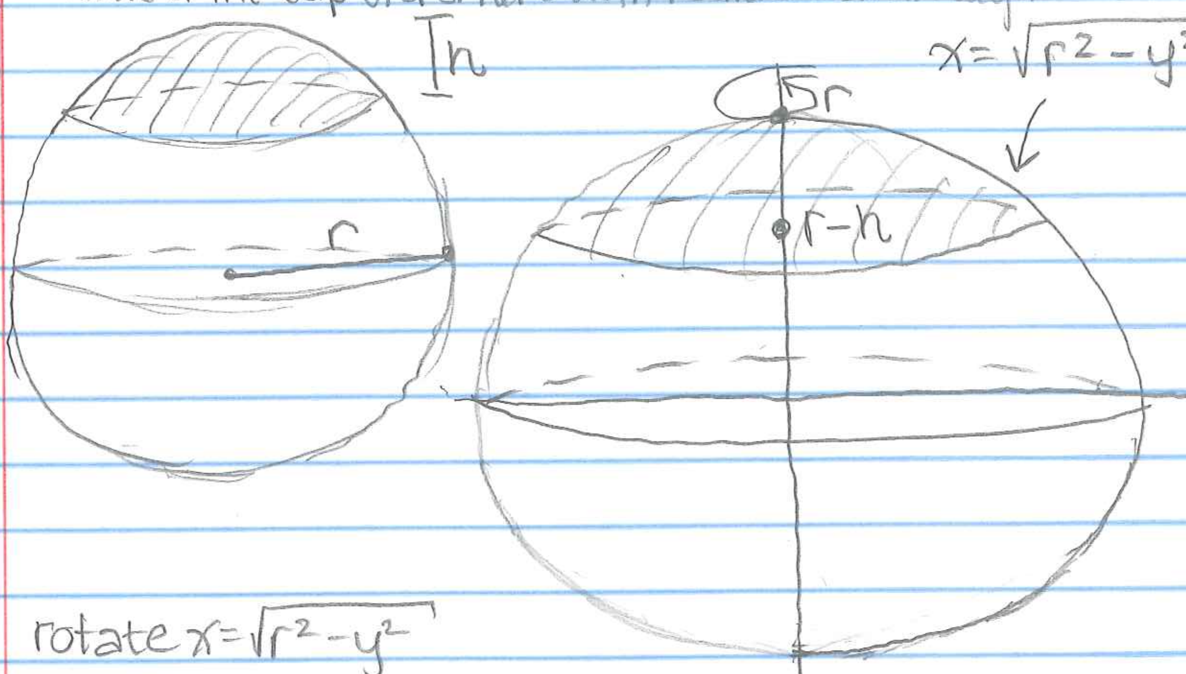
- rotates around an axis parallel to y-axis
- $1-y^2$ is radius



Region bounded by $y=-1, y=1, x=1-y^2$ rotated about the y-axis.

Volume of the cap of a sphere with radius r and height h

49.



rotate $x=\sqrt{r^2-y^2}$
about the y-axis

$$\text{Volume} = \int_{r-h}^{r} \pi\left(\sqrt{r^2-y^2}\right)^2 dy$$

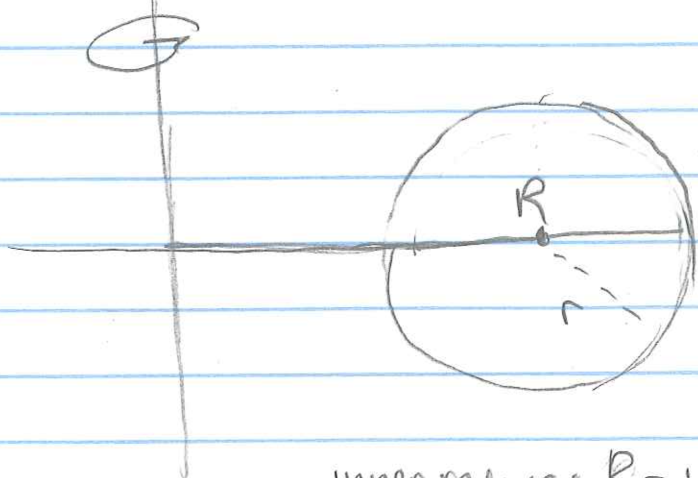
$$= \int_{r-h}^{r} \pi(r^2-y^2)\,dy = \pi\left(r^2y-\frac{y^3}{3}\right)\Big|_{r-h}^{r}$$

$$= \pi\left(r^3-\frac{r^3}{3}-r^2(r-h)+\frac{(r-h)^3}{3}\right)$$

$$= \pi\left(\frac{2r^3}{3}-r^3+r^2h+\frac{r^3}{3}-r^2h+rh^2-\frac{h^3}{3}\right)$$

$$= \pi\left(rh^2-\frac{h^3}{3}\right)$$

61.



$x = \sqrt{r^2 - y^2} + R$ (right half of circle)

$x = -\sqrt{r^2 - y^2} + R$ (left half of circle)

inner radius: $R - \sqrt{r^2 - y^2}$

Outer radius: $R + \sqrt{r^2 - y^2}$

$$\text{Volume} = \int_{-r}^{r} \pi\left(R + \sqrt{r^2 - y^2}\right)^2 - \pi\left(R - \sqrt{r^2 - y^2}\right)^2 \, dy$$

$$= \int_{-r}^{r} \pi\left(R^2 + 2R\sqrt{r^2 - y^2} + (r^2 - y^2) - R^2 + 2R\sqrt{r^2 - y^2} - (r^2 - y^2)\right)$$

$$= \int_{-r}^{r} 4R\pi\sqrt{r^2 - y^2}$$

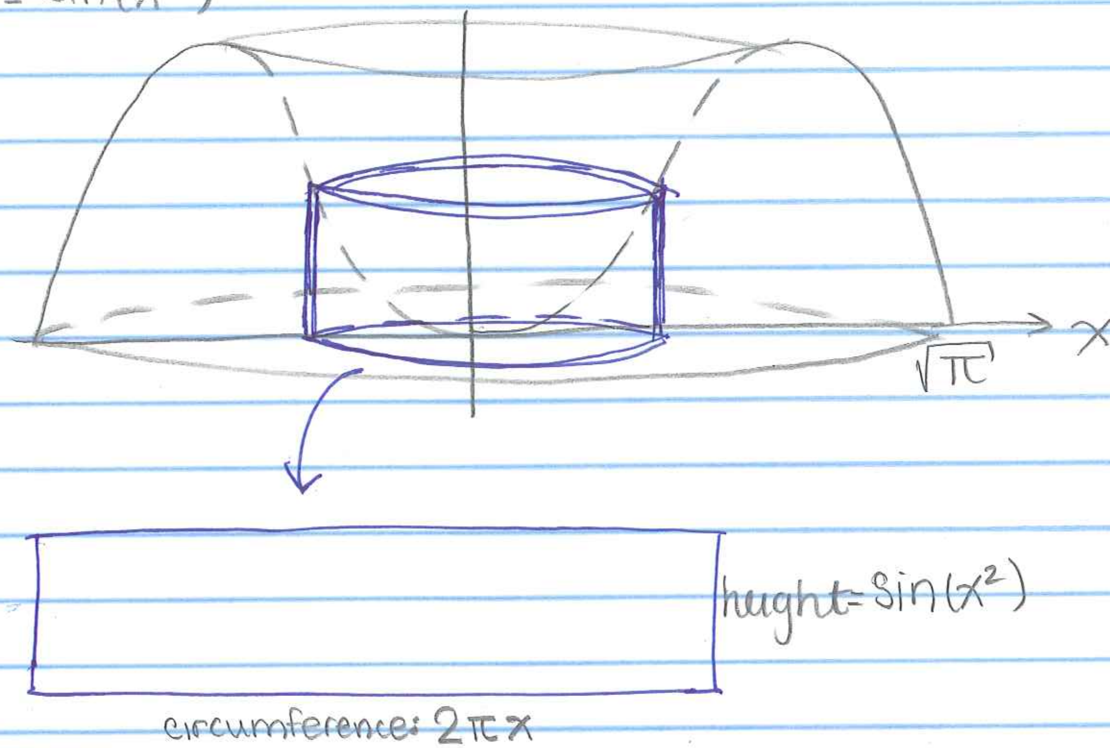
(b) $$\int_{-r}^{r} 4R\pi\sqrt{r^2 - y^2}\, dy = 4R\pi \int_{-r}^{r} \underbrace{\sqrt{r^2 - y^2}}^{\text{half circle of radius } r} dy$$

$$= 4R\pi\left(\frac{\pi r^2}{2}\right) = 2\pi^2 R r^2$$

6.3: 2, 8, 14, 38, 45, 48

6.3

2. $y = \sin(x^2)$



$$\text{Vol} = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\,dx \qquad u = x^2$$

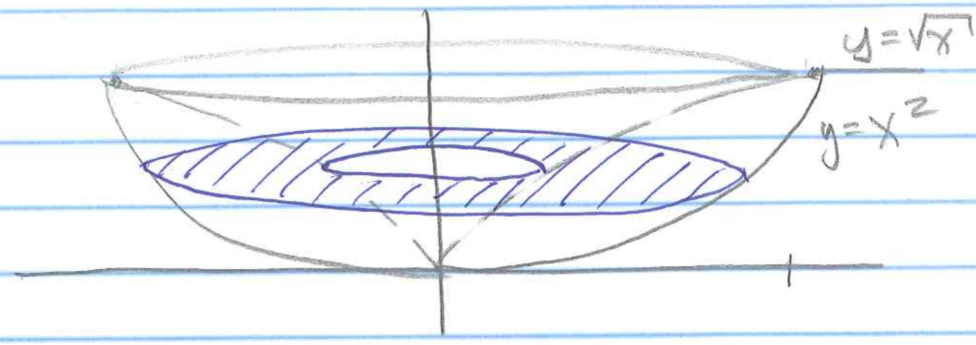
$$du = 2x\,dx$$

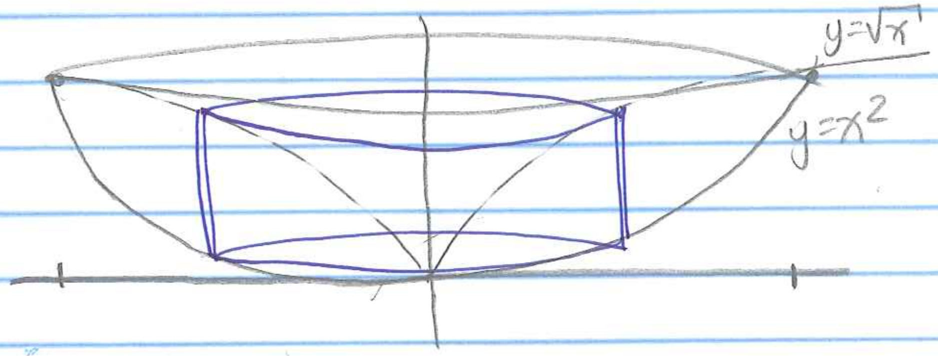
$$= \int_{x=0}^{x=\sqrt{\pi}} \pi \sin u \, du = -\pi \cos(u) \Big|_{x=0}^{x=\sqrt{\pi}}$$

$$= -\pi \cos(x^2) \Big|_0^{\sqrt{\pi}} = -\pi\cos(\pi) + \pi\cos(0)$$

$$= \pi + \pi = \boxed{2\pi}$$

8.





Disk/Washer	Cylindrical Shells
outer radius: $\sqrt{y}$	height: $\sqrt{x}-x^2$
inner radius: y^2	circumference: $2\pi x$
$\int_0^1 \pi(\sqrt{y})^2-\pi(y^2)^2\,dy$	$\int_0^1 2\pi x(\sqrt{x}-x^2)\,dx$
$=\int_0^1 \pi y-\pi y^4\,dy$	$=\int_0^1 2\pi(x^{3/2}-x^3)\,dx$
$=\pi\frac{y^2}{2}-\pi\frac{y^5}{5}\Big\vert_0^1$	$=2\pi\left(\frac{2}{5}x^{5/2}-\frac{x^4}{4}\right)\Big\vert_0^1$
$=\frac{\pi}{2}-\frac{\pi}{5}=\boxed{\frac{3\pi}{10}}$	$=2\pi\left(\frac{2}{5}-\frac{1}{4}\right)=2\pi\left(\frac{3}{20}\right)=\boxed{\frac{3\pi}{10}}$

14. $x+y=3,\ x=4-(y-1)^2$ rotate about x-axis

$x=3-y,\ x=4-(y-1)^2$ *in terms of y*

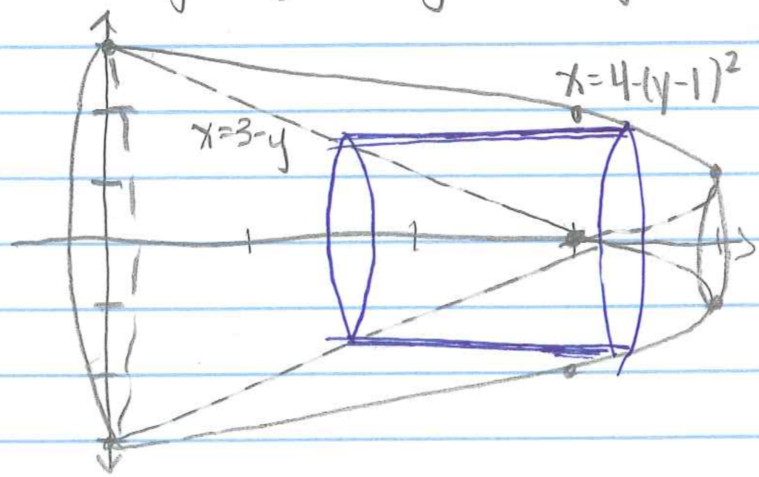
Intersection Points:

$3-y=4-(y-1)^2$

$-y=1-(y^2-2y+1)$

$0=3y-y^2$

$0=y(3-y) \Rightarrow y=0$ OR $y=3$



Cylinders Width/height: $4-(y-1)^2-(3-y)$

Circumference: $2\pi y$

$$\int_0^3 2\pi y\left(4-(y-1)^2-(3-y)\right)dy$$

$$=2\pi\int_0^3 y\left(4-y^2+2y-1-3+y\right)dy=2\pi\int_0^3 3y^2-y^3\,dy$$

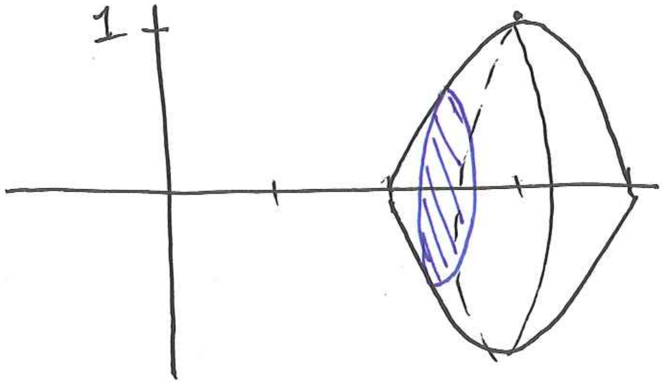
$$=2\pi\left(y^3-\frac{y^4}{4}\right)\Big|_0^3$$

$$=2\pi\left(27-\frac{81}{4}\right)=\left(\frac{108-81}{2}\right)\pi=\boxed{\frac{27\pi}{2}}$$

38. $y=-x^2+6x-8,\ y=0$; about the x-axis

x	0	1	2	3	4	5
y	-8	-3	0	1	0	-3

$\Big\}$ Values of $y=-x^2+6x-8$ to draw the graph



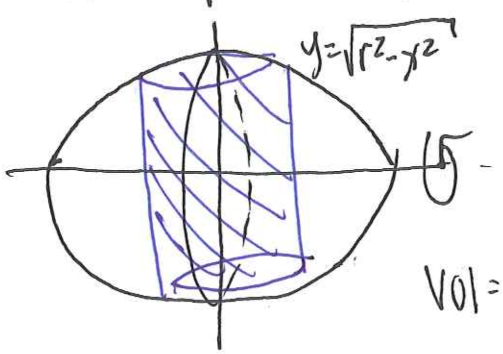
* use disk method
  - if you were to use shells, you would need the integral in terms of y. Messy. (But possible)

$$\int_2^4 \pi(-x^2+6x-8)^2\,dx=\int_2^4 \pi(x^4-12x^3+52x^2-96x+64)\,dx$$

$$=\pi\left(\frac{x^5}{5}-12\frac{x^4}{4}+52\frac{x^3}{3}-\frac{96x^2}{2}+64x\right)\Bigg|_2^4$$

$$=\boxed{\frac{16\pi}{15}}$$

45. A sphere is $y=\sqrt{r^2-x^2}$ rotated about the x-axis.



Cylindrical shells:

Height: $2\sqrt{r^2-x^2}$

Circumference: $2\pi x$

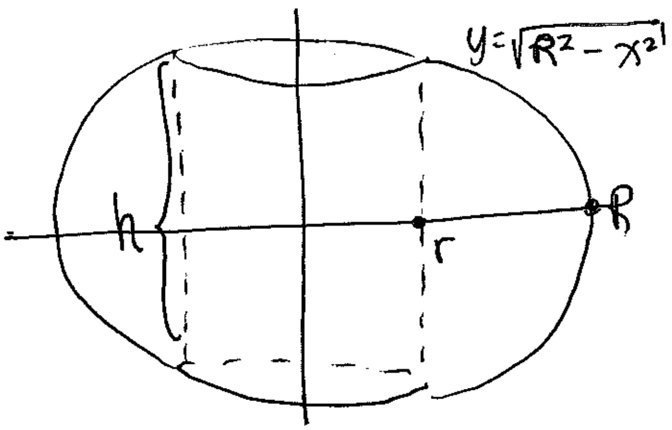
$$\text{Vol}=\int_0^r 2\pi x\cdot 2\sqrt{r^2-x^2}\,dx$$

u-sub: $u=r^2-x^2,\ du=-2x\,dx$

$$=\int_{x=0}^{x=r}-2\pi\sqrt{u}\,du=-\pi\frac{4}{3}u^{3/2}\Bigg|_{x=0}^{x=r}=-\pi\frac{4}{3}(r^2-x^2)^{3/2}\Bigg|_0^r=\frac{4\pi r^3}{3}$$

48. (a) Professor Guess: The right one.

(b) Same procedure as in 45, w/ different bounds



Vol: $\displaystyle\int_r^R 4\pi x\sqrt{R^2-x^2}\,dx$

$$= -2\pi\cdot\frac{2}{3}(R^2-x^2)^{3/2}\Big|_r^R$$

$$= \frac{4\pi}{3}(R^2-r^2)^{3/2}$$

This doesn't tell us much. We need to put this in terms of h.

Notice that $h=2\sqrt{R^2-r^2} \Rightarrow \frac{h^2}{4}=R^2-r^2$

So, $\frac{4\pi}{3}(R^2-r^2)^{3/2} = \frac{4\pi}{3}\left(\frac{h^2}{4}\right)^{3/2} = \underline{\underline{\frac{\pi}{6}\cdot h^3}}$

This implies the rings have the <u>same</u> volume. Whoa.