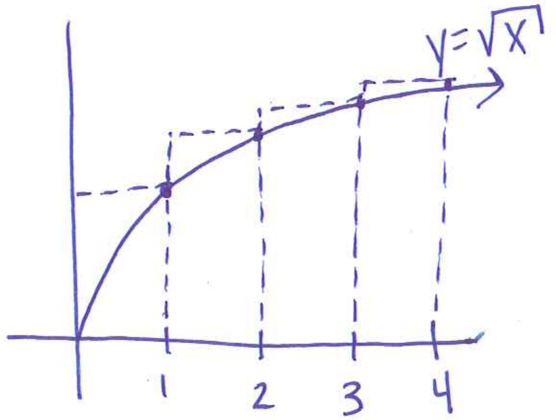
1. All parts of this question concern the function $y = \sqrt{x}$ on the interval $0 \le x \le 4$

(a) (5 pts) Use four rectangles with right endpoints to approximate the area under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 4$. Is this an over or under approximation?



$\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4}$

$= 3 + \sqrt{2} + \sqrt{3}$

Overapproximation

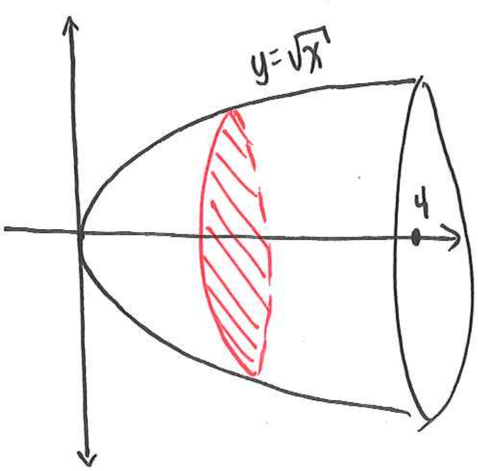
(b) (5 pts) Find the actual area under the curve using Part 2 of the Fundamental Theorem of Calculus.

$$\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$$

(c) (5 pts) Find the average value of the function $y = \sqrt{x}$ on the interval $0 \leq x \leq 4$.

$$\frac{1}{4-0} \cdot \int_0^4 \sqrt{x}\, dx = \frac{1}{4} \cdot \frac{2}{3} \cdot x^{3/2} \Big|_0^4$$
$$= \frac{1}{6} x^{3/2} = \frac{1}{6} \cdot 4^{3/2}$$
$$= \frac{4}{3}$$

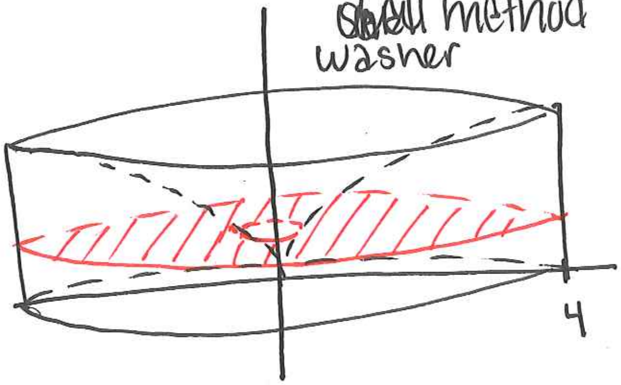
(d) (5 pts) Find the volume of the solid obtained by rotating the region enclosed by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ about the x-axis.



$$\int_0^4 \pi (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx$$
$$= \pi \frac{x^2}{2} \Big|_0^4$$
$$= \pi \frac{16}{2} = \boxed{8\pi}$$

(e) (5 pts) Find the volume of the solid obtained by rotating the region enclosed by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ about the y-axis.

~~shell~~ method
washer



Inner radius: $x = y^2$

Outer radius: 4

$$\int_0^2 \pi\left[(4)^2 - (y^2)^2\right] dy$$

$$= \pi \int_0^2 (16 - y^4)\, dy$$

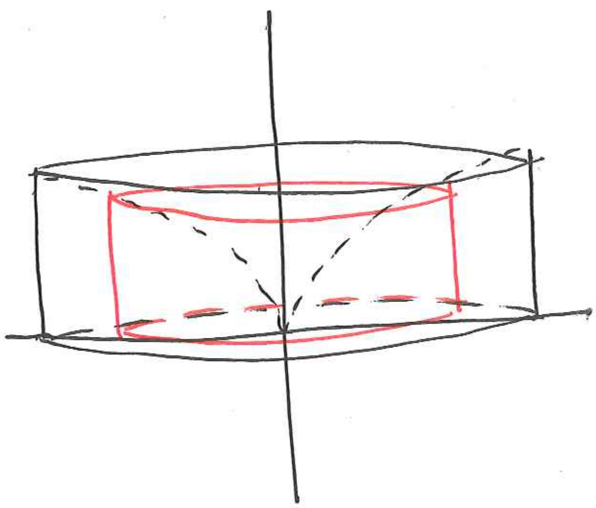
$$= \pi \left(16y - \frac{y^5}{5}\right)\Big|_0^2$$

$$= \pi \left(32 - \frac{32}{5}\right)$$

$$= \pi\left(\frac{4 \cdot 32}{5}\right) = \frac{128}{5}\pi$$

OR

cylindrical shell method



$$\int_0^4 2\pi x \sqrt{x}\, dx$$

$$= \int_0^4 2\pi x^{3/2} dx$$

$$= 2\pi \cdot \frac{2}{5} x^{5/2} \Big|_0^4$$

$$= \frac{4\pi}{5} \cdot 32 = \frac{128}{5}\pi$$

2. In this question you will state both parts of the Fundamental Theorem of Calculus:

Suppose $f(x)$ is continuous on $[a, b]$.

(3 pts) Part 1: If $g(x) = \int_a^x f(t)dt$ then

$$g'(x) = f(x).$$

(3 pts) Part 2:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

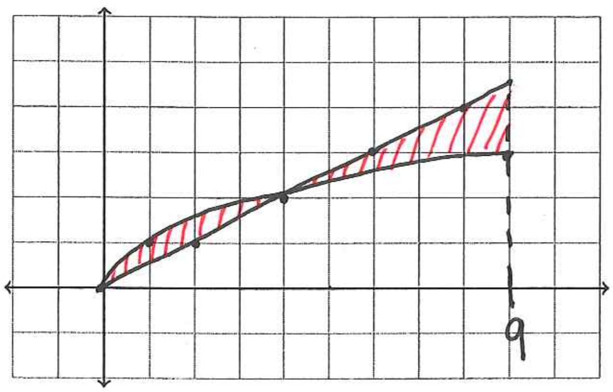
3. (3 pts) If $g(x) = \int_2^x \frac{1}{(t-1)(t+2)^2}\, dt$ find $g'(x)$

$$g'(x) = \frac{1}{(x-1)(x+2)^2}$$

Use FTC part I.

4. In this problem you will find the area of the region enclosed by the curves $y = \sqrt{x}$ and $y = \frac{x}{2}$ on the interval $0 \le x \le 9$

(a) (0 pts) Sketch the region that the curves enclose.



(b) (10 pts) Find the area of the region enclosed by the curves.

$$\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)dx + \int_4^9 \left(\frac{x}{2} - \sqrt{x}\right)dx = \text{Area}$$

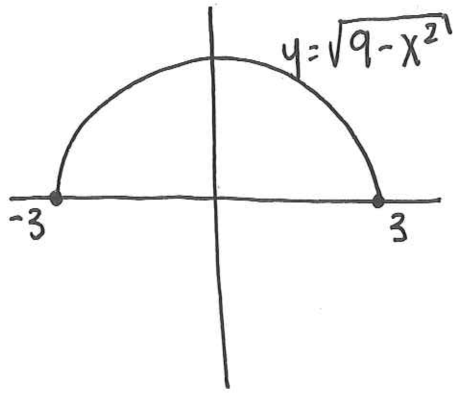
$$= \left(\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right)\Big|_0^4 + \left(\frac{x^2}{4} - \frac{2}{3}x^{3/2}\right)\Big|_4^9$$

$$= \frac{2}{3}(4)^{3/2} - \frac{(4)^2}{4} + \frac{(9)^2}{4} - \frac{2}{3}(9)^{3/2} - \frac{(4)^2}{4} + \frac{2}{3}(4)^{3/2}$$

$$= \frac{16}{3} - 4 + \frac{81}{4} - \frac{54}{3} - 4 + \frac{16}{3}$$

$$= -\frac{22}{3} - 8 + \frac{81}{4} = \frac{59}{12}$$

5. (10 pts) Use the Arc Length formula to find the circumference of a circle with equation $x^2 + y^2 = 9$



Arc length of semicircle:

$$\int_{-3}^{3} \sqrt{1 + \left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\, dx$$

$$= \int_{-3}^{3} \sqrt{1 + \frac{x^2}{9-x^2}}\, dx$$

$$= \int_{-3}^{3} \frac{\sqrt{9-x^2+x^2}}{\sqrt{9-x^2}}\, dx$$

trig-sub: $x = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$

$$= \int_{-3}^{3} \frac{3}{\sqrt{9-x^2}}\, dx$$

$$= \int_{x=-3}^{x=3} \frac{9\cos\theta\, d\theta}{\sqrt{9-(3\sin\theta)^2}}$$

$$= \int_{x=-3}^{x=3} 3\, d\theta$$

$$= 3\theta \Big|_{x=-3}^{x=3}$$

$\frac{x}{3} = \sin\theta \Leftrightarrow \sin^{-1}\left(\frac{x}{3}\right) = \theta$

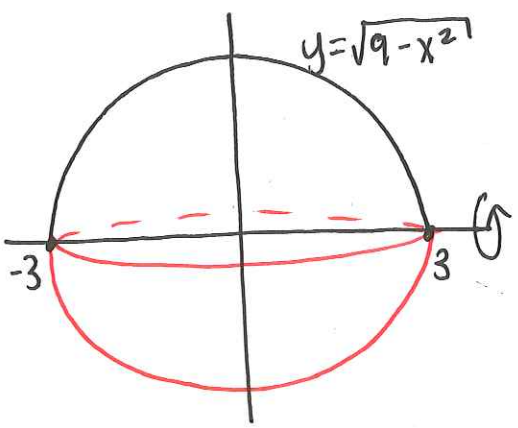
$$= 3\sin^{-1}\left(\frac{x}{3}\right)\Big|_{-3}^{3}$$

$$= 3\sin^{-1}(1) - 3\sin^{-1}(-1)$$

$$= \frac{3\pi}{2} - \left(-\frac{3\pi}{2}\right) = 3\pi$$

For full circumference, multiply by 2: $2 \cdot 3\pi = \boxed{6\pi}$

6. (10 pts) Use the Surface Area formula to find the surface area of a sphere with radius $r = 3$



$$\int_{-3}^{3} 2\pi\sqrt{9-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\,dx$$

$$= \int_{-3}^{3} 2\pi\sqrt{(9-x^2)\left(1+\frac{x^2}{9-x^2}\right)}\,dx$$

$$= 2\pi\int_{-3}^{3}\sqrt{9-x^2+x^2}\,dx$$

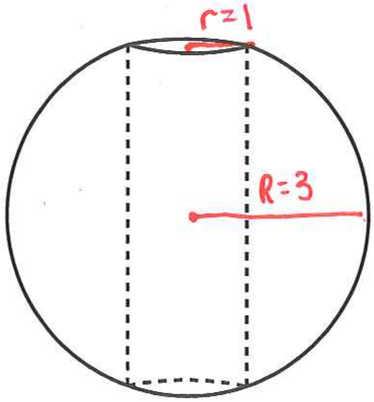
$$= 2\pi\int_{-3}^{3} 3\,dx$$

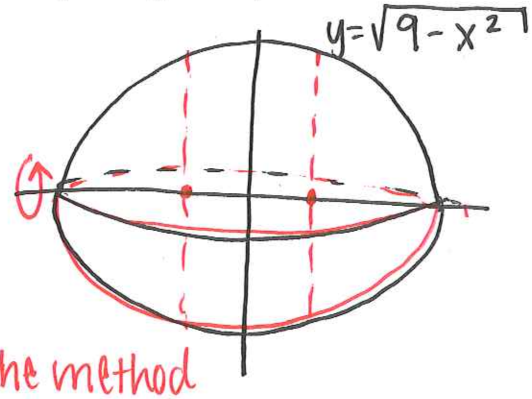
$$= 6\pi x\Big|_{-3}^{3}$$

$$= 6\pi(3-(-3))$$

$$= \boxed{36\pi}$$

7. (10 pts) Find the volume of a napkin ring (or bead) obtained by drilling a hole with radius $r = 1$ through the center of a wooden sphere of radius $R = 3$ (see image below).





use the method of cylindrical shells.

$$\int_1^3 2\pi x\sqrt{9-x^2}\,dx \qquad \begin{array}{l} u = 9-x^2 \\ du = -2x\,dx \end{array}$$

$$= \int_{x=1}^{x=3} -\pi\sqrt{u}\,du$$

$$= -\frac{2}{3}\pi u^{3/2}\Big|_{x=1}^{x=3}$$

$$= -\frac{2}{3}\pi\left(9-x^2\right)^{3/2}\Big|_1^3$$

$$= \frac{2}{3}\pi(9-1)^{3/2}$$

$$= \frac{2}{3}\pi(8)^{3/2} = \boxed{\frac{32\sqrt{2}}{3}\pi}$$

8. Determine whether the following integrals converge or diverge. Evaluate those that converage.

(a) (5 pts) $\int_1^\infty \frac{1}{x^4}\,dx$ $= \lim_{t\to\infty} \int_1^t \frac{1}{x^4}\,dx$

$$= \lim_{t\to\infty} -\frac{1}{3x^3}\Big|_1^t$$

$$= \lim_{t\to\infty} \Big(\underbrace{-\frac{1}{3t^3}}_{\to 0} + \frac{1}{3}\Big)$$

$$= \boxed{\frac{1}{3}}$$ converges

(b) (5 pts) $\int_0^\infty \frac{1}{\sqrt{1+x}}\,dx$ $= \lim_{t\to\infty} \int_0^t \frac{1}{\sqrt{1+x}}\,dx$

$$= \lim_{t\to\infty} 2\sqrt{1+x}\Big|_0^t$$

$$= \lim_{t\to\infty} \underbrace{2\sqrt{1+t}}_{\to\infty} - 2$$

DIVERGES

(c) (5 pts) $\int_0^\infty x \cdot e^{-x^2}\, dx$ $= \lim_{t\to\infty} \int_0^t xe^{-x^2} dx$

$u = -x^2$

$du = -2x\,dx$

$= \lim_{t\to\infty} -\frac{1}{2}\int_0^t e^u du$

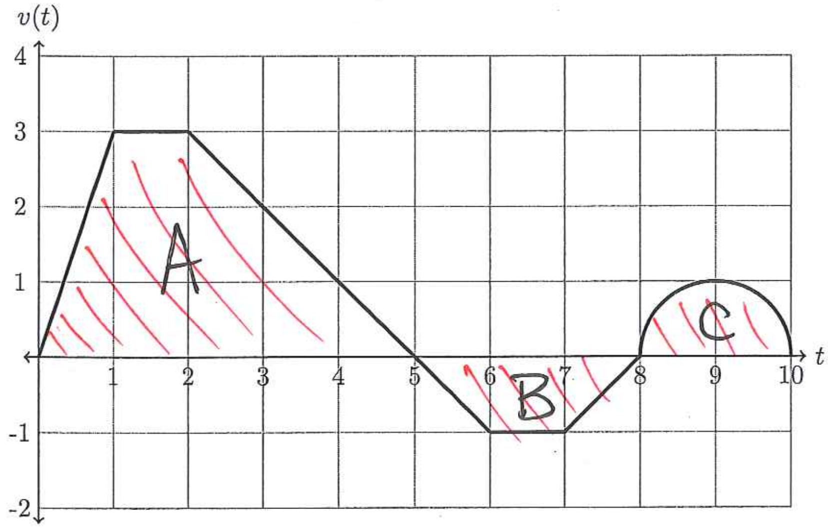
$= \lim_{t\to\infty} -\frac{1}{2} e^u \Big|_0^t$

$= \lim_{t\to\infty} -\frac{1}{2} e^{-x^2} \Big|_0^t$

$= \lim_{t\to\infty} \underbrace{-\frac{1}{2} e^{-t^2}}_{\to 0} + \frac{1}{2}$

$= \boxed{\frac{1}{2}}$ CONVERGES

9. Below is the graph of the velocity function $v(t)$ of a particle moving along a straight line on the time interval $0 \le t \le 10$.



(a) (4 pts) Find the displacement of the particle on the time interval $0 \le t \le 10$.

Displacement $= A - B + C$

$$\left[\frac{1}{2}(1)(3) + (1)(3) + \frac{1}{2}(3)(3)\right] - \left[2\right] + \left[\frac{1}{2}\pi\right]$$

$$= \boxed{7 + \frac{\pi}{2}}$$

(b) (4 pts) Find the total distance traveled by the particle on the time interval $0 \leq t \leq 10$.

Total distance = A+B+C

$$9 + 2 + \frac{\pi}{2} = \boxed{11 + \frac{\pi}{2}}$$

(c) (3 pts) Find the acceleration of the particle on the time interval $2 < t < 6$.

acceleration = slope

$$= \boxed{-1}$$

10. Evaluate the following integrals

(a) (5 pts) $\int_1^3 x^3 \ln x \, dx$ int by parts

$$u = \ln x \qquad dv = x^3 dx$$
$$du = \frac{1}{x} \qquad v = \frac{x^4}{4}$$

$$= \frac{x^4}{4} \cdot \ln x \Big|_1^3 - \int_1^3 \frac{x^3}{4} dx$$

$$= \frac{x^4}{4} \cdot \ln x \Big|_1^3 - \frac{x^4}{16}\Big|_1^3$$

$$= \frac{3^4}{4} \cdot \ln 3 - \frac{3^4}{16} + \frac{1}{16} = \frac{81}{4} \cdot \ln 3 - \frac{80}{16}$$

$$= \boxed{\frac{81}{4} \cdot \ln 3 - 5}$$

(b) (5 pts) $\int \frac{5x^3 - 2x^2 - 2x + 1}{x^4 - x^2} dx$ partial fraction decomp

$$\frac{5x^3 - 2x^2 - 2x + 1}{x^4 - x^2} = \frac{5x^3 - 2x^2 - 2x + 1}{x^2(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$5x^3 - 2x^2 - 2x + 1 = A(x^3 - x) + B(x^2 - 1) + C(x^3 - x^2) + D(x^3 + x^2)$$

$5 = A + C + D$ $A = 2$

$-2 = B - C + D$ $B = -1$

$-2 = -A$ $C = 2$

$1 = -B$ $D = 1$

$$\int \frac{2}{x} + \frac{-1}{x^2} + \frac{2}{x+1} + \frac{1}{x-1} \, dx = \boxed{2\ln x + \frac{1}{x} + 2\ln(x+1) + \ln(x-1) + C}$$

(c) (5 pts) $\int \frac{3x}{x^2+x-2}\,dx$ partial fraction decomp.

$$\frac{3x}{x^2+x-2} = \frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$3x = A(x-1) + B(x+2)$$

$3 = A + B$ $\quad B = 1$

$0 = -A + 2B$ $\quad A = 2$

$$\int \frac{2}{x+2} + \frac{1}{x-1}\,dx = \boxed{2\ln(x+2) + \ln(x-1) + C}$$

(d) (5 pts) $\int \arctan(x)\,dx$ note on notation: $\arctan(x) = \tan^{-1}(x) \neq \frac{1}{\tan(x)}$

int by parts

$u = \arctan x$ $\quad dv = 1\,dx$

$du = \frac{1}{1+x^2}dx$ $\quad v = x$

u-sub:

$u = 1+x^2$

$du = 2x\,dx$

$$= x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$$= x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \boxed{x\arctan x - \frac{1}{2}\ln(1+x^2) + C}$$

(e) (5 pts) $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

trig-sub
$x = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$

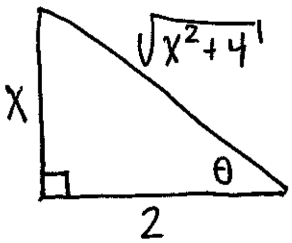
$$= \int \frac{(2\tan\theta)^3}{\sqrt{4\tan^2\theta + 4}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int 8\tan^3\theta \sec\theta\, d\theta = 8\int \tan^2\theta (\tan\theta\sec\theta)\, d\theta$$

u-sub
$u = \sec\theta$
$du = \tan\theta\sec\theta$

$$= 8\int (\sec^2\theta - 1)(\tan\theta\sec\theta)\, d\theta$$

$$= 8\int u^2 - 1\, du = 8\left(\frac{u^3}{3} - u\right) = 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right)$$



$$= 8\left(\frac{(x^2+4)^{3/2}}{3\cdot 2^3} - \frac{(x^2+4)^{1/2}}{2}\right) + C$$

$$= \boxed{\frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C}$$

(f) (5 pts) $\int \sec x \cdot \tan x \cdot \sin(\sec x)\, dx$

u-Substitution
$u = \sec x$
$du = \sec x \tan x\, dx$

$$= \int \sin u\, du$$

$$= -\cos u + C = \boxed{-\cos(\sec x) + C}$$

11. (10 pts) Find the arc length of the curve $y = \ln(\sec x)$ on the interval $0 \leq x \leq \pi/4$

$$y' = \frac{\sec x \tan x}{\sec x} = \tan x$$

$$\text{Arc Length} = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec x \, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$= \ln\left|\frac{2}{\sqrt{2}} + 1\right| - \ln|1 + 0|$$

$$= \boxed{\ln\left|\frac{2}{\sqrt{2}} + 1\right|}$$

12. (10 pts) Find the surface area obtained by rotating the curve $y = \sqrt{1+e^x}$, $0 \le x \le 1$ about the x-axis.

Hint: one of these equalities will be needed $1 + e^x + \frac{e^{2x}}{4} = (1 + \frac{e^x}{2})^2$ $\quad$ $4 + 4e^x + e^{2x} = (2 + e^x)^2$

$$y' = \frac{e^x}{2\sqrt{1+e^x}}$$

$$\int_0^1 2\pi\sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\,dx$$

$$= \int_0^1 2\pi\sqrt{1+e^x+\frac{e^{2x}}{4}}\,dx$$

$$= \int_0^1 2\pi\sqrt{(1+\frac{e^x}{2})^2}\,dx$$

$$= \int_0^1 2\pi\left(1+\frac{e^x}{2}\right)dx$$

$$= 2\pi\left(x+\frac{e^x}{2}\right)\Big|_0^1$$

$$= 2\pi\left(1+\frac{e}{2}-\frac{1}{2}\right)$$

$$= 2\pi\left(\frac{1}{2}+\frac{e}{2}\right)$$

$$= \boxed{\pi + \pi\cdot e}$$