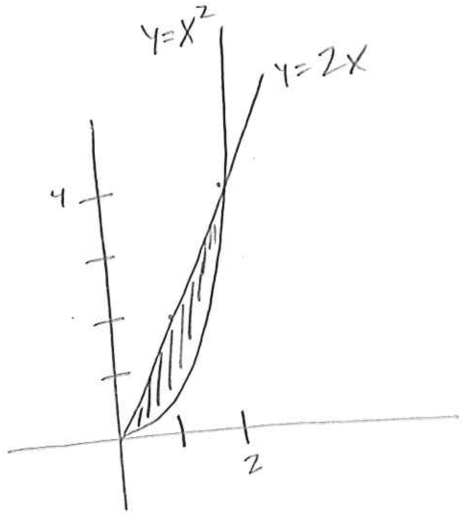
1. All three parts of this problem concern the region bounded by the curves $y = 2x$ and $y = x^2$.

(a) (6 pts) Find the area of the region, i.e. find the area enclosed by the curves $y = 2x$ and $y = x^2$.

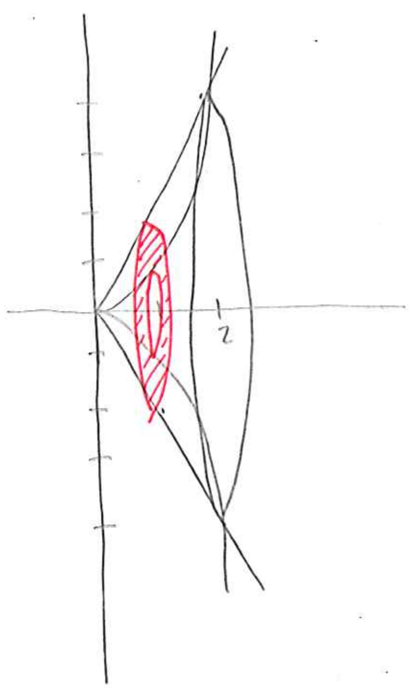


$$\text{Area} = \int_0^2 2x - x^2 \, dx$$

$$= x^2 - \frac{x^3}{3} \Big|_0^2$$

$$= 4 - \frac{8}{3} = 4/3$$

(b) (6 pts) Use the disk or washer method to find the volume of the solid obtained by rotating the region about the x-axis.



outer radius : $2x$

inner radius : x^2

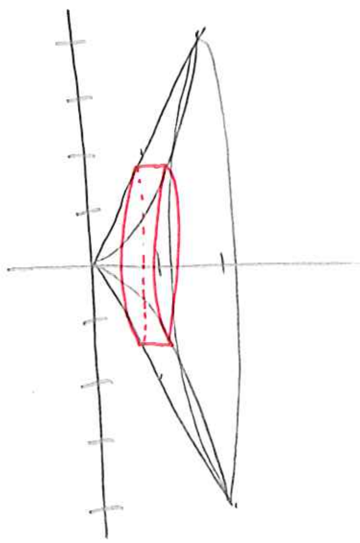
$$\text{Vol} = \int_0^2 \pi (2x)^2 - \pi (x^2)^2 \, dx$$

$$= \pi \int_0^2 4x^2 - x^4 \, dx = \pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^2$$

$$= \pi \left(\frac{4}{3} \cdot 8 - \frac{1}{5} \cdot 32 \right)$$

$$= \pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{64}{15} \pi$$

(c) (6 pts) Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region about the x-axis.



width of shell : $\sqrt{y} - \frac{1}{2}y$

$$\text{Vol} = \int_0^4 2\pi y \left(\sqrt{y} - \tfrac{1}{2}y\right) dy$$

$$= 2\pi \int_0^4 y^{3/2} - \tfrac{1}{2}y^2 \, dy$$

$$= 2\pi \left(\tfrac{2}{5}y^{5/2} - \tfrac{1}{2}\cdot\tfrac{1}{3}y^3\right)\Big|_0^4$$

$$= 2\pi\left(\tfrac{2}{5}4^{5/2} - \tfrac{1}{6}4^3\right) = \frac{64}{15}\pi$$

2. (6 pts) Use integration by parts with $u = x$ and $dv = e^x\, dx$ to evaluate the integral $\displaystyle\int_0^1 x e^x\, dx$.

$u = x \qquad dv = e^x dx$

$du = 1 \cdot dx \qquad v = e^x$

$$\int_0^1 xe^x dx = xe^x \Big|_0^1 - \int_0^1 e^x dx$$

$$= xe^x \Big|_0^1 - e^x \Big|_0^1$$

$$= e - e + 1 = \boxed{1}$$

3. Mickey and Minnie are racing. Mickey Mouse has velocity function $v(t) = t^2$.

Minnie Mouse has velocity function $v(t) = \frac{1}{3}(t-1)^3 + \frac{1}{3}$.

(a) (6 pts) Find the average velocity of Minnie Mouse between $t = 0$ and $t = 2$.

$$\text{Avg} = \frac{1}{2-0}\int_0^2 \frac{1}{3}(t-1)^3 + \frac{1}{3}\, dt = \frac{1}{6}\int_0^2 (t-1)^3 + 1\, dt$$

$$= \frac{1}{6}\left(\frac{(t-1)^4}{4} + t\right)\Big|_0^2$$

$$= \frac{1}{6}\left(\frac{1}{4} + 2 - \frac{1}{4}\right) = \boxed{\frac{1}{3}}$$

Don't need to expand. If you did, you get:

$$\frac{1}{2}\int_0^2 \frac{1}{3}(t^3 - 3t^2 + 3t - 1) + \frac{1}{3}\, dt = \frac{1}{2}\int_0^2 \frac{t^3}{3} - t^2 + t\, dt$$

$$= \frac{1}{2}\left(\frac{t^4}{12} - \frac{t^3}{3} + \frac{t^2}{2}\right)\Big|_0^2 = \frac{1}{2}\left(\frac{4}{3} - \frac{8}{3} + 2\right)$$

$$= \boxed{\frac{1}{3}}$$

(b) (6 pts) Who is ahead after 2 seconds, Mickey or Minnie? What is the distance between the two mice after 2 seconds?

$$\int_0^2 t^2 dt = \frac{t^3}{3}\Big|_0^2 = \frac{8}{3} \quad \text{(position of Mickey)}$$

$$\int_0^2 \frac{1}{3}(t-1)^3 + \frac{1}{3} = \frac{1}{3}\frac{(t-1)^4}{4} + \frac{1}{3}t\Big|_0^2 = \frac{1}{12} + \frac{2}{3} - \frac{1}{12} = \frac{2}{3} \quad \text{(position of Minnie)}$$

Mickie is ahead after 2 seconds.

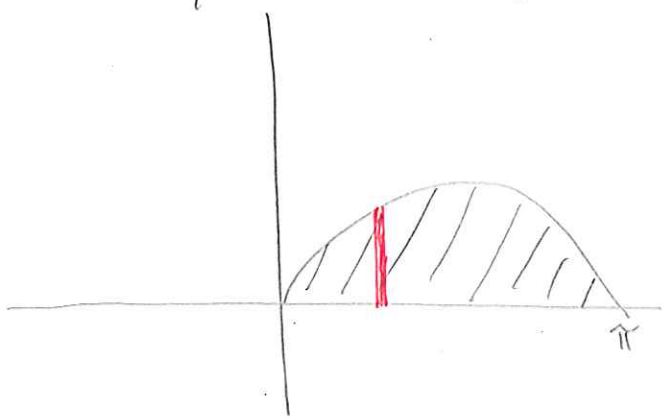
This distance between the two is:

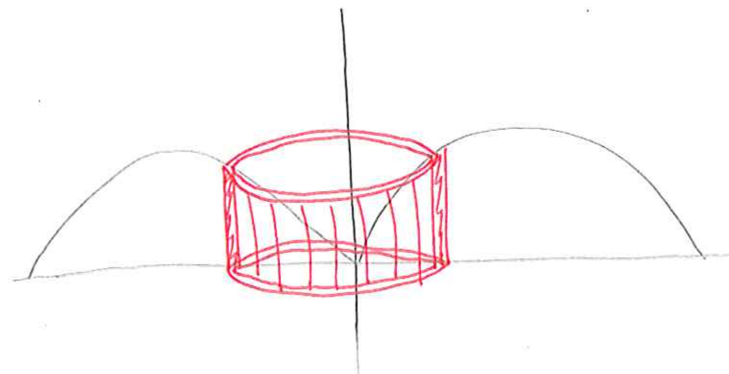
$$\frac{8}{3} - \frac{2}{3} = \frac{6}{3} = \boxed{2}$$

4. (12 pts) Find the volume of the solid obtained by rotating the region bounded by $y = \sin x$ and $y = 0$ and between $x = 0$ and $x = \pi$ about the y-axis.

note: the region is the first hump of the sine curve

use cylindrical shells





$$\text{Vol} = \int_0^{\pi} 2\pi x \cdot \sin x \, dx = 2\pi \int_0^{\pi} x \cdot \sin x \, dx$$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$$= 2\pi \left(-x\cos x + \int_0^{\pi} \cos x \, dx \right)$$

$$= 2\pi \left(-x\cos x + \sin x \right) \Big|_0^{\pi}$$

$$= 2\pi \left(-\pi \cos(\pi) + \sin \pi \right) - 2\pi \left(0 + \sin(0) \right)$$

$$= 2\pi^2$$

5. Evaluate the following indefinite integrals

(a) (6 pts) $\int \sin^3 x \cdot \cos^5 x \, dx$ $= \int \sin x \cdot \sin^2 x \cdot \cos^5 x \, dx$

$= \int \sin x \,(1-\cos^2 x)\cos^5 x \, dx$ $\quad u = \cos x \quad du = -\sin x \, dx$

$= -\int (1-u^2)\,u^5 \, du$

$= -\int u^5 - u^7 \, du$

$= -\left(\frac{u^6}{6} - \frac{u^8}{8}\right) + C$

$= \boxed{-\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C}$

Alt sol'n

$= \int \cos x \sin^3 x \cos^4 x \, dx$

$= \int \cos x \sin^3 x\,(1-\sin^2 x)^2 \, dx$

$u = \sin x \quad du = \cos x \, dx$

$= \int u^3 (1-u^2)^2 \, du$

$= \int u^3 (1 - 2u^2 + u^4)\, du$

$= \int u^3 - 2u^5 + u^7 \, du$

$= u^4/4 - 2u^6/6 + u^8/8 + C$

$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C$

(b) (6 pts) $\int x \cdot e^{x^2} \, dx$

substitution: $u = x^2$

$du = 2x \, dx$

$\int x e^{x^2} dx = \int \frac{1}{2} e^u \, du = \frac{1}{2} e^u + C = \boxed{\frac{1}{2} e^{x^2} + C}$

(c) (6 pts) $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

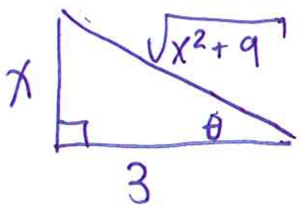
trig substitution; $x = 3\tan\theta \quad dx = 3\sec^2\theta$

$$\int \frac{27\tan^3\theta}{\sqrt{9\tan^2\theta + 9}} 3\sec^2\theta = \int \frac{27\tan^3\theta \cdot 3\sec^2\theta}{3\sec\theta}\,d\theta = \int 27\tan^3\theta \cdot \sec\theta\,d\theta$$

$$= 27\int (\tan\theta\sec\theta)\tan^2\theta\,d\theta = 27\int (\tan\theta\sec\theta)(\sec^2\theta - 1)\,d\theta$$

u-sub: $u = \sec\theta \quad du = \sec\theta\tan\theta\,d\theta$

$$= 27\int (u^2-1)\,du = 27\left(\frac{u^3}{3} - u + C\right) = 27\left(\frac{\sec^3\theta}{3} - \sec\theta + C\right)$$



$$= 27\left(\frac{(x^2+9)^{3/2}}{3\cdot 3^3} - \frac{\sqrt{x^2+9}}{3} + C\right)$$

$$= \boxed{\frac{(x^2+9)^{3/2}}{3} - 9\sqrt{x^2+9} + C}$$

(d) (6 pts) $\int \arcsin(x)\,dx$

note on notation: $\arcsin(x) = \sin^{-1}(x)$

Integration by Parts

$u = \arcsin x \qquad dv = 1\,dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x\,dx$

u-sub: $u = 1 - x^2$
$du = -2x\,dx$

$$= x \cdot \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arcsin x - \int \left(-\frac{1}{2}\right)\frac{1}{\sqrt{u}}\,du = x\arcsin x + \frac{1}{2}\cdot 2\sqrt{u} + C$$

$$= \boxed{x\arcsin x + \sqrt{1-x^2} + C}$$

(e) (6 pts) $\int e^x \cdot \sin(x)\, dx$ integration by parts

$u = e^x \quad dv = \sin x$

$du = e^x \quad v = -\cos x\, dx$

$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$

$u = e^x \quad dv = \cos x\, dx$

$du = e^x dx \quad v = \sin x$

$\Rightarrow 2 \int e^x \sin x\, dx = e^x \sin x - e^x \cos x$

$\int e^x \sin x\, dx = \boxed{\dfrac{e^x \sin x - e^x \cos x}{2} + C}$

EASIER WAY (f) (6 pts) $\int \frac{2x}{\sqrt{x^2-1}}\, dx$

u-substitution:

$u = x^2 - 1$

$du = 2x\, dx$

$\int \frac{2x}{\sqrt{x^2-1}}\, dx = \int \frac{1}{\sqrt{u}}\, du = 2\sqrt{u} + C$

$= \boxed{2\sqrt{x^2-1} + C}$

HARDER WAY

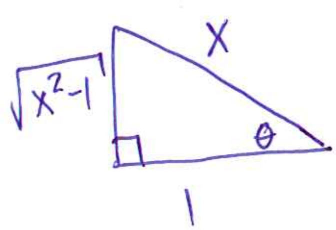
trig substitution: $x = \sec\theta$

$dx = \sec\theta \tan\theta\, d\theta$

$\int \frac{2\sec\theta \cdot \sec\theta \tan\theta\, d\theta}{\sqrt{\sec^2\theta - 1}}$

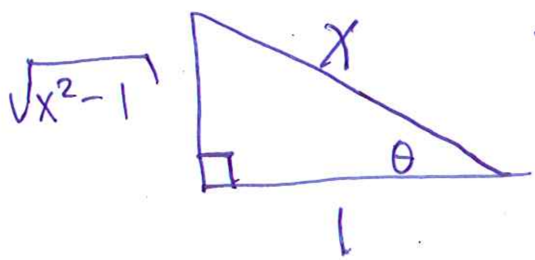
$= \int 2\sec^2\theta\, d\theta = 2\tan\theta + C$

$= \boxed{2\sqrt{x^2-1} + C}$



(g) (6 pts) $\int \frac{1}{2x\sqrt{x^2-1}}\, dx$ $\quad x = \sec\theta$, $\quad dx = \sec\theta\tan\theta\, d\theta$

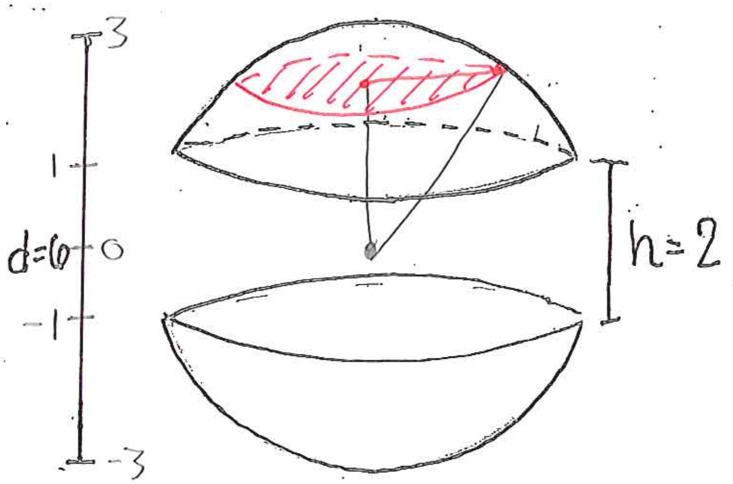
$$= \int \frac{\sec\theta\tan\theta\, d\theta}{2\sec\theta\sqrt{\sec^2\theta - 1}} = \int \frac{1}{2}\, d\theta = \frac{1}{2}\theta + C = \boxed{\frac{1}{2}\sec^{-1}x + C}$$

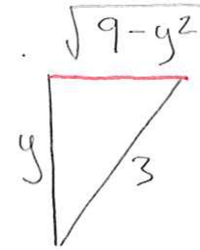


$x = \sec\theta \Rightarrow \sec^{-1} x = \theta$

6. (10 pts) Scientists are concerned about global warming and have decided to remove the topical region of the earth. Assume the earth is a sphere with diameter $d = 6$ and that the tropical region has total height $h = 2$. Find the volume of the portion of the earth remaining after the tropics have been removed.

note: the remaining region consists of a top cap and a bottom cap.





What's left of the Earth with the tropics removed.
(NH Survives)

$$A(y) = \pi\left(\sqrt{9-y^2}\right)^2 = \pi(9-y^2)$$

because top and bottom cap have same Vol.

$$\text{Vol (caps)} = \int_{-3}^{-1} A(y)\,dy + \int_{1}^{3} A(y)\,dy = 2\int_{1}^{3} A(y)\,dy$$

$$= 2\int_{1}^{3} \pi(9-y^2)\,dy = 2\pi\left(9y - \frac{y^3}{3}\right)\Big|_1^3$$

$$= 2\pi\left(27 - \frac{27}{3}\right) - 2\pi\left(9 - \frac{1}{3}\right)$$