**1.** [12 points] **Multiple choice.** Circle the correct answer for each question.

(a) Consider the following:

$$\int \frac{\ln(\ln(x))}{x\ln(x)}\,dx = \int u\,du.$$

What substitution did we make?

$u = \ln(\ln(x))$

A. $u = \ln(x)$

B. $u = x\ln(x)$

$du = \frac{1}{\ln(x)} \frac{1}{x} dx$

C. $u = 1/\ln(x)$

D. $u = \ln(\ln(x))$

E. $u = e^x$

chain rule

(b)

$$\int_{-\pi}^{\pi} x^2 \sin(x)\cos(x)\,dx =$$

A. 0, because $x^2\sin(x)\cos(x)$ is odd

B. 0, because $x^2\sin(x)\cos(x)$ is even

C. $2\int_0^{\pi} x^2\sin(x)\cos(x)\,dx$, because $x^2\sin(x)\cos(x)$ is odd

D. $2\int_0^{\pi} x^2\sin(x)\cos(x)\,dx$, because $x^2\sin(x)\cos(x)$ is even

E. none of the above

(c) Which differentiation rule gives rise to $u$-substitution?

A. Chain rule

B. Power rule

C. Product rule

D. Integration by parts

E. Fundamental Theorem of Calculus

(d) Consider the integral

$$\int 3x^2 \sin(x^3)\, dx.$$

What substitution should we make to find this integral?

A. $u = x^2$
B. $u = x^3$
C. $u = 3x^2$
D. $u = \sin(x)$
E. $u = \sin(x^3)$

(e) If we use integration by parts on the integral

$$\int x^3 \sin(x)\, dx,$$

then we should pick $u$ and $dv$ to be:

A. $u = x^3$ and $dv = dx$
B. $u = x^3 \sin(x)$ and $dv = dx$
C. $u = x^3$ and $dv = \cos(x)\, dx$
D. $u = \cos(x)$ and $dv = x^3\, dx$
E. $u = x^3$ and $dv = \sin(x)\, dx$

(f) Consider the region enclosed by $y = x^2$ and $x = y^2$. Rotate this region around the $y$-axis to get a solid. Set up the integral for volume of the solid using cylindrical shells.

A. $V = \int_0^1 2\pi y(\sqrt{y} - y^2)\, dy$
B. $V = \int_0^1 \pi(\sqrt{x} - x^2)^2\, dx$
C. $V = \int_0^1 \pi(\sqrt{y} - y^2)^2\, dy$
D. $V = \int_0^1 2\pi y(y^2 - \sqrt{y})\, dy$
E. $V = \int_0^1 2\pi x(\sqrt{x} - x^2)\, dx$

$V = \int_0^1 2\pi x\, h(x)\, dx$

$h(x) = \sqrt{x} - x^2$

2. [6 points] Find $\int_{-1}^{0} 3x^2\sqrt{x^3+1}\,dx$.

$X=-1 \rightsquigarrow u=0$

Let $u=x^3+1$. Then $du = 3x^2\,dx$. Change bounds:

$X=0 \rightsquigarrow u=1$

$$\int_{-1}^{0} 3x^2\sqrt{x^3+1}\,dx = \int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \boxed{\frac{2}{3}}$$

3. [6 points] Find $\int \ln(x)\,dx$. IBP

Let $u=\ln(x)$ $\quad v=x$

$du = \frac{1}{x}dx$ $\quad dv = dx$

$$\int \ln(x)\,dx = x\ln(x) - \int x\,\frac{1}{x}\,dx = x\ln(x) - \int dx = \boxed{x\ln(x) - x + C}$$

4. [6 points] Find $\int \tan(x)\,dx$. u-sub.

Let $u=\cos(x)$. Then $du = -\sin(x)\,dx$.

$$\int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{1}{u}\sin(x)\,dx = -\int \frac{1}{u}\,du = -\ln(u) + C$$

$$= \boxed{-\ln(\cos(x)) + C}$$

5. [6 points] Derive the formula for integration by parts.

product rule: $d(uv) = u\,dv + v\,du$

integrate both sides: $uv = \int u\,dv + \int v\,du$

rearrange: $\int u\,dv = uv - \int v\,du$.

6. [8 points] Find $\int x^3 e^{x^2}\, dx$.

u-sub: let $w = x^2$. Then $dw = 2x\,dx$.

$$\int x^3 e^{x^2} dx = \frac{1}{2}\int x^2 e^w\, dw \underset{\uparrow}{=} \frac{1}{2}\int w e^w\, dw$$

Backsubstitute

IBP: $u = w$ $\quad v = e^w$

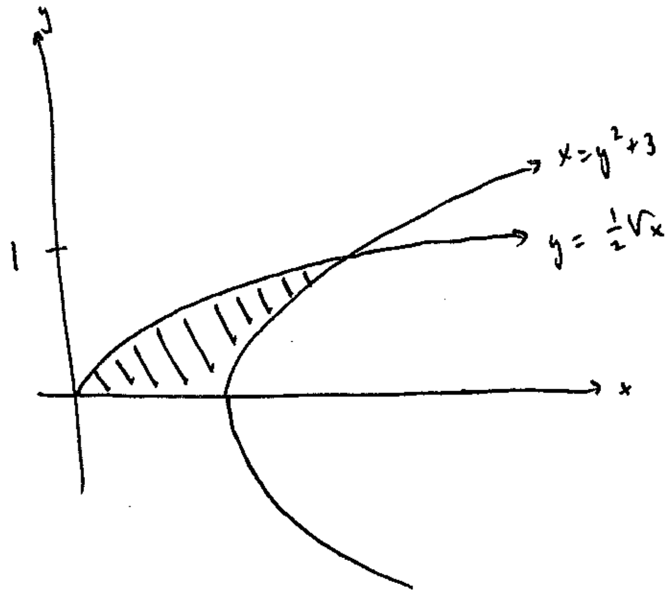
$du = dw$ $\quad dv = e^w\, dw$

$$\frac{1}{2}\int we^w\, dw = \frac{1}{2}\left( we^w - \int e^w dw \right) = \frac{1}{2}\left(we^w - e^w\right) + C = \frac{e^w}{2}(w-1) + C$$

$$= \boxed{\frac{e^{x^2}}{2}(x^2-1) + C}$$

**7.** [12 points] Find the area of the region in the first quadrant enclosed by the three curves

$$y = 0 \qquad y = \frac{1}{2}\sqrt{x} \qquad x = y^2 + 3.$$




Easier to integrate with respect to y.

$y = \frac{1}{2}\sqrt{x}$

$2y = \sqrt{x}$

$4y^2 = x$

Point of intersection: $y^2 + 3 = 4y^2$

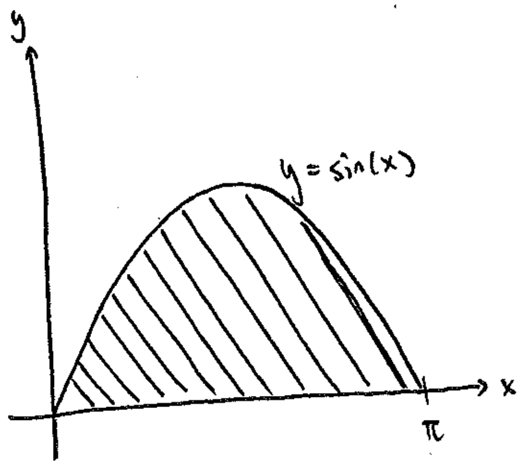
$3 = 3y^2$

$1 = y^2$

$y = \pm 1$

$$A = \int_0^1 y^2 + 3 - 4y^2 \, dy = \int_0^1 3 - 3y^2 \, dy = \left[3y - y^3\right]_0^1 = 3 - 1 = \boxed{2}$$

8. [12 points] Consider the region bounded by the curves $y = \sin(x)$ and $y = 0$ as pictured below. Rotate this region about the $y$-axis to form a solid. Find the volume of this solid.




Shells: $V = \int_0^\pi 2\pi x \, h(x)\,dx$

$= \int_0^\pi 2\pi x \sin(x)\,dx = 2\pi \int_0^\pi x \sin(x)\,dx$

IBP: $u = x$ $\quad v = -\cos(x)$

$du = dx$ $\quad dv = \sin(x)\,dx$

$$2\pi \int_0^\pi x \sin(x)\,dx = 2\pi \left( -x\cos(x)\Big|_0^\pi + \overset{0}{\cancel{\int_0^\pi \cos(x)\,dx}} \right)$$

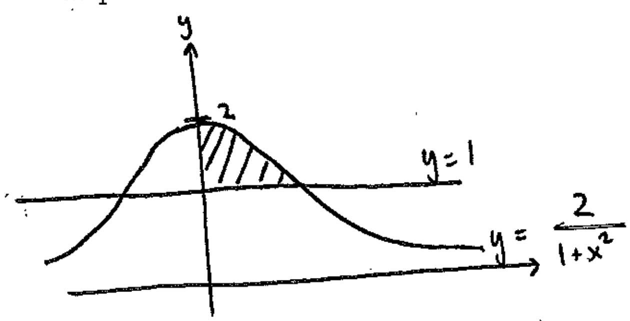
$$= 2\pi\left(-\pi \cos(\pi)\right)$$

$$= \boxed{2\pi^2}.$$

**9.** [18 points] Consider the region in the first quadrant enclosed by the three curves

$$y = \frac{2}{1+x^2} \qquad\qquad y = 1 \qquad\qquad x = 0,$$

as pictured below. By rotating this region around the $y$-axis, we form a solid.




(a) Find the volume of the solid using disks/washers (slices).

$V = \int_1^2 A(y)\,dy.$ To find $A(y)$, we need to solve $y = \frac{2}{1+x^2}$ for $x$:

$$y(1+x^2) = 2 \Rightarrow 1+x^2 = \frac{2}{y} \Rightarrow x^2 = \frac{2}{y} - 1 \Rightarrow x = \sqrt{\frac{2}{y} - 1}.$$

$$A(y) = \pi x^2 = \pi\left(\frac{2}{y} - 1\right).$$

$$V = \int_1^2 \pi\left(\frac{2}{y} - 1\right) dy = \pi \int_1^2 \frac{2}{y}\,dy - \pi \int_1^2 dy = 2\pi \int_1^2 \frac{1}{y}\,dy - \pi$$

$$= 2\pi\left[\ln(y)\right]_1^2 - \pi = \boxed{2\pi \ln(2) - \pi}$$

(b) Find the volume of the solid using cylindrical shells.

$V = \int_a^b 2\pi x \, h(x)\, dx$. To find bounds, we need to find where $y=1$ intersects

$y = \frac{2}{1+x^2}$. Set them equal and solve: $1 = \frac{2}{1+x^2} \Rightarrow 1+x^2 = 2 \Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$. Therefore the bounds of the integral are $0$ and $1$.

The graph suggests $h(x) = \frac{2}{1+x^2} - 1$.

$$V = \int_0^1 2\pi x \left( \frac{2}{1+x^2} - 1 \right) dx = 2\pi \int_0^1 \frac{2x}{1+x^2}\, dx - 2\pi \int_0^1 x\, dx$$

To find the first of these two integrals, we'll need u-substitution: $u = x^2+1$ $du = 2x\,dx$

How do the bounds change: $x=0 \rightsquigarrow u=1$ $x=1 \rightsquigarrow u=2$.

$$V = 2\pi \int_1^2 \frac{1}{u}\, du - 2\pi \left[ \frac{x^2}{2} \right]_0^1$$

$$= 2\pi \left[ \ln(u) \right]_1^2 - 2\pi \left( \frac{1}{2} \right)$$

$$= \boxed{2\pi \ln(2) - \pi}$$

**10.** [14 points] Choose and circle one of the following techniques for integration:

$u$-substitution integration by parts

(a) Explain a strategy for applying your chosen technique.

u-substitution: Choose $u = g(x)$ so that

1.) $g(x)$ is ~~the~~ a complicated part of the integral

2.) both $g(x)$ and $g'(x)$ appear in the integral, and you can replace $g'(x)dx$ with $du$

3.) if you still have $x$ left over in the integral, you can solve for $x$ in terms of $u$ and backsubstitute.

(b) Illustrate this strategy on an integral of your choosing. Solve your integral using your chosen technique. The integral you choose should not come from this exam.

integration by parts: If your integral is the product of two functions, ~~choose~~ choose one of those functions to be $u$ and the other $v'$. Choose $u$ to be whatever function comes first in the following list:

Logarithmic
Inverse Trig
Algebraic
Trig
Exponential

**Bonus:** Find

$$\int \arcsin(x)\,dx.$$

IBP: let $u = \arcsin(x)$ $\qquad v = x$

$$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad dv = dx$$

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

To get this second integral, we make a <u>w-substitution</u> $w = 1-x^2$. So $dw = -2x\,dx$.

$$\int \arcsin(x)\,dx = x\arcsin(x) + \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw = x\arcsin(x) + \frac{1}{2}\int w^{-1/2}\,dw$$

$$= x\arcsin(x) + w^{1/2} + C$$

$$= \boxed{x\arcsin(x) + \sqrt{1-x^2} + C}$$