1. [12 points] **Multiple choice.** Circle the correct answer for each question.

(a) Find the limit

$$\lim_{x\to 1} \frac{x^{20}-1}{x-1}. \leftarrow \text{type } \frac{0}{0}$$

A) 0
B) 1
C) 20
D) 19
E) -1

l'Hôpital: $= \lim_{x\to 1} \frac{20x^{19}}{1} = 20.$

Alternatively: $x^{20}-1 = (x-1)(x^{19}+x^{18}+x^{17}+\dots+x^{1}+x+1).$

(b) The antiderivative of $-3\sin(x)$ is

A) $3\cos(x)+C$
B) $-3\cos(x)+C$
C) $\cos(3x)+C$
D) $3\cos(3x)+C$
E) $-\cos(3x)+C$

(c) Evaluate the integral

$$\int_1^e \frac{2}{x}\,dx.$$

A) 1
B) $2e-2$
C) $2-2/e^2$
D) $2/e^2-2$
E) 2

$= 2\int_1^e \frac{dx}{x}$

$= 2\ln(x)\Big|_1^e = 2\ln(e) - 2\ln(1)$

$= 2$

(d) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{6x}$ is an indeterminate form of type

A) ∞^∞
B) 0^∞
C) ∞^0
D) 1^∞
E) 0^0

$\frac{3}{x} \to 0$ as $x \to \infty$

$6x \to \infty$ as $x \to \infty$

(e) Evaluate the integral

$$\int_0^{\pi/4} \sec(x)\tan(x)\,dx.$$

A) $\sqrt{2}-1$
B) 1
C) $\sqrt{2}/2-1$
D) $\pi/6$
E) $\sqrt{2}$

$= \sec(x)\Big|_0^{\pi/4}$

$= \frac{1}{\cos(\pi/4)} - \frac{1}{\cos(0)}$

$= \sqrt{2} - 1$

(f) The antiderivative of $(3-5x)\sqrt{x}$ is

A) $-\sqrt{x}/(3-5x)^2+C$
B) $(3x-5x^2/2)(2x^{3/2}/3)+C$
C) $2x^{3/2}-2x^{5/2}+C$
D) $-15\sqrt{x}/2+3x^{-1/2}/2+C$
E) $-5\sqrt{x}+C$

$(3-5x)\sqrt{x} = 3x^{1/2} - 5x^{3/2}$.

$\int (3x^{1/2} - 5x^{3/2})\,dx$

$= \frac{2}{3}\cdot 3x^{3/2} - \frac{2}{5}\cdot 5x^{5/2} + C$

$= 2x^{3/2} - 2x^{5/2} + C$

2. [3 points] Write the following sum using $\sum$ notation:

$$3\ln(3) + 4\ln(4) + 5\ln(5) + 6\ln(6) + 7\ln(7).$$

$$\sum_{i=3}^{7} i \ln(i)$$

3. [5 points] Suppose $\int_0^3 f(x)\,dx = 2$, $\int_3^7 2f(x)\,dx = 3$ and $\int_7^0 g(x)\,dx = 3$. Find

$$\int_0^7 (4f(x) + g(x))\,dx.$$

$$\int_0^3 f(x)dx = 2 \Rightarrow \int_0^3 4f(x)\,dx = 8$$

$$\int_3^7 2f(x) = 3 \Rightarrow \int_3^7 4f(x)dx = 6$$

$$\int_7^0 g(x)dx = 3 \Rightarrow \int_0^7 g(x)dx = -3$$

$$\int_0^7 (4f(x) + g(x))dx$$

$$= \int_0^7 4f(x)\,dx + \int_0^7 g(x)dx$$

$$= 8+6 \quad -3$$

$$= \boxed{11}$$

4. [5 points] Using the Fundamental Theorem of Calculus, evaluate the integral

$$\int_0^2 (2x^3 - 1)\,dx.$$

$$= \frac{1}{2}x^4 - x \Big|_0^2 = \frac{1}{2}2^4 - 2$$

$$= \frac{1}{2}16 - 2 = 8 - 2 = \boxed{6}$$

5. [5 points] Using the Fundamental Theorem of Calculus, evaluate the integral

$$\int_0^\pi \left(e^x - \sin(x)\right) dx.$$

$$= e^x + \cos(x) \Big|_0^\pi = e^\pi + \cos(\pi) - e^0 - \cos(0)$$

$$= e^\pi - 1 - 1 - 1$$

$$= \boxed{e^\pi - 3}$$

6. [8 points] Evaluate the limit

Let $y = \lim_{x \to \infty} x^{\frac{1}{\sqrt{x}}}.$

type $\frac{\infty}{\infty}$

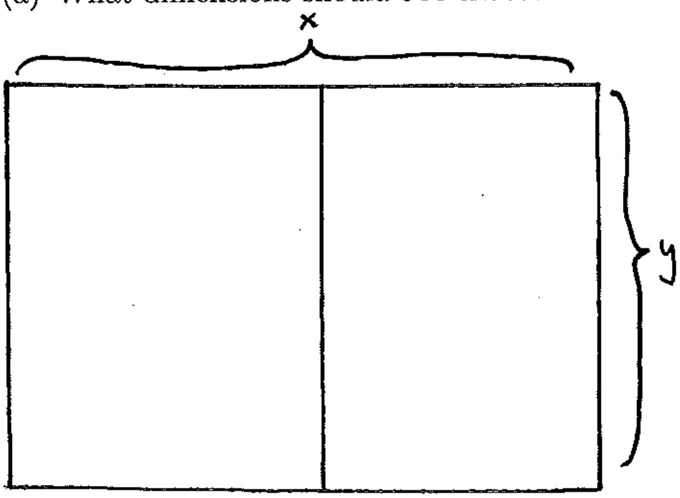
$$\ln(y) = \lim_{x\to\infty} \ln\left(x^{1/\sqrt{x}}\right) = \lim_{x\to\infty} \frac{1}{\sqrt{x}} \ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$$

l'Hôpital: $$= \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{1}{x} \frac{2\sqrt{x}}{1} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

Therefore $y = e^{\ln(y)} = e^0 = \boxed{1}$

7. [10 points] Farmer Joe wants to enclose an area of 150 square yards with fencing, then divide it in half with a fence parallel to one of the sides of the rectangle as shown in the picture. Each yard of fencing costs \$2.

(a) What dimensions should Joe choose so as to minimize the cost of the fence?



area $= xy = 150$

cost $= C = (2x+3y)\,2$ \$

to be minimized.

Eliminate variable: $y = \dfrac{150}{x}$.

$$C(x) = 4x + 6y = 4x + 6\left(\frac{150}{x}\right) = 4x + \frac{900}{x}$$

$$C'(x) = 4 - \frac{900}{x^2}. \qquad C'(x) = 0 \implies 4 - \frac{900}{x^2} = 0 \implies x^2 = \frac{900}{4} = 225$$

$$\implies x = 15. \quad \text{Then } y = \frac{150}{x} = \frac{150}{15} = 10.$$

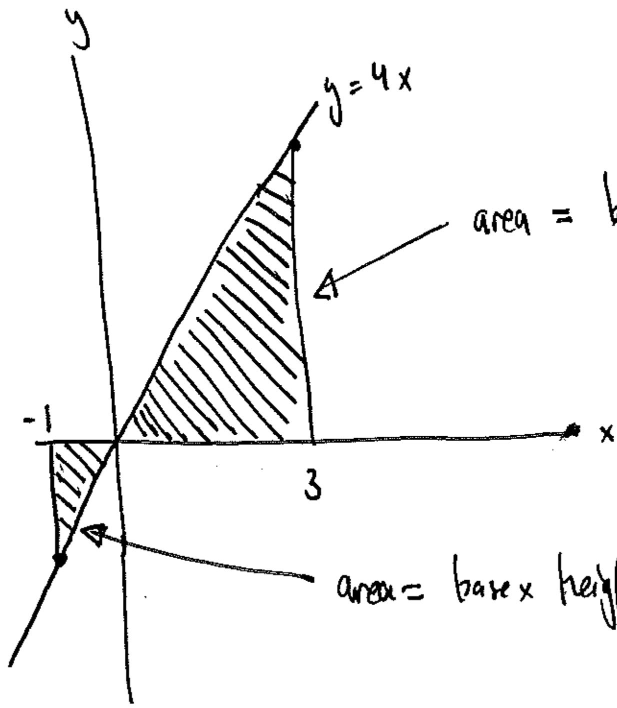
Dimensions: $\boxed{15 \text{ yards} \times 10 \text{ yards}}$

(b) What is the minimum cost?

$$\text{cost} = C = (2x+3y)\,2 = (2\cdot 15 + 3\cdot 10)\,2 = \boxed{120\ \$}$$

8. [15 points] In this problem you will compute the integral $\int_{-1}^{3} 4x\,dx$ in three different ways.

(a) Evaluate this integral by interpreting it in terms of area. You could start by drawing the graph of the function $y = 4x$ on the interval $[-1, 3]$.



$$\text{area} = \frac{\text{base} \times \text{height}}{2} = \frac{3 \cdot 12}{2} = 18$$

$$\text{area} = \text{base} \times \text{height} = \frac{1 \cdot 4}{2} = 2.$$

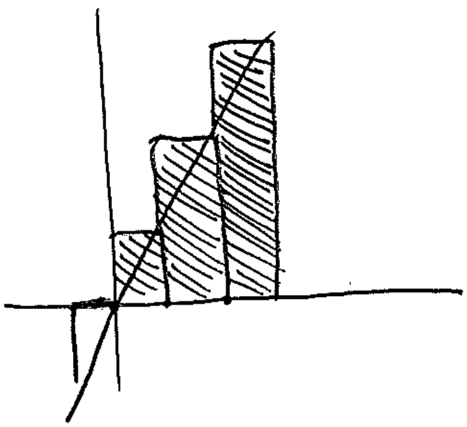
$$\int_{-1}^{3} 4x\,dx = \text{area}(\triangle) - \text{area}(\triangledown) = 18 - 2 = \boxed{16}$$

above x-axis

below x-axis

(b) Approximate this integral using a right endpoint Riemann sum with four terms ($n = 4$). Is this estimate an over-approximation or under-approximation?

$$\Delta x = \frac{b-a}{n} = \frac{3-(-1)}{4} = 1.$$



Riemann sum:

$$f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x$$

$$= 0 + 4 + 8 + 12$$

$$= \boxed{24.}$$

Comparing this with part a, it is an over-approximation.

(c) Evaluate this integral using the Fundamental Theorem of Calculus.

$$\int_{-1}^{3} 4x\,dx = 2x^2 \Big|_{-1}^{3} = 2\cdot 3^2 - 2\cdot(-1)^2$$

$$= 18 - 2$$

$$= \boxed{16.}$$

9. [12 points] A water balloon is dropped off the roof of a 400 foot tall building. The acceleration due to gravity is -32 feet/second^2.

(a) How long does it take before the water balloon hits the ground?

$a(t) = -32 \ \text{ft}/s^2.$

$v(t) = -32t + C \ \text{ft/s}. \qquad v(0) = 0 \Rightarrow C = 0.$

$s(t) = -16t^2 + D \ \text{ft}. \qquad s(0) = 400 \Rightarrow D = 400.$

$s(t) = -16t^2 + 400.$

Solve for t when $s(t) = 0$:

$-16t^2 + 400 = 0 \Rightarrow 16t^2 = 400 \Rightarrow t^2 = 25 \Rightarrow t = 5.$

$\boxed{5 \text{ seconds.}}$

(b) What is the velocity of the balloon immediately before it explodes?

$v(5) = -32 \cdot 5 = \boxed{-160 \ \text{ft/s}.}$

10. [15 points] The following questions deal with the Fundamental Theorem of Calculus.

(a) State both parts of the Fundamental Theorem of Calculus.

Part 1. Suppose $f(t)$ is continuous on $[a,b]$. For $a \le x \le b$, define

$g(x) = \int_a^x f(t)\,dt$. Then $g(x)$ is differentiable and $g'(x) = f(x)$.

Part 2: Suppose $f(x)$ is continuous on $[a,b]$. Let $F(x)$ be any antiderivative for $f(x)$.

Then $\int_a^b f(x)\,dx = F(x)\Big|_a^b$.

(b) Find the derivative of the function $g(x) = \int_0^x (t^2 + \sin(t))\,dt$ in two different ways:

(i) Apply part one of the Fundamental Theorem.

$\boxed{x^2 + \sin(x).}$

(ii) Use part two of the Fundamental Theorem to find a formula for $g(x)$, then take the derivative.

$$g(x) = \int_0^x (t^2 + \sin(t))\,dt = \frac{1}{3}t^3 - \cos(t)\Big|_0^x = \frac{1}{3}x^3 - \cos(x) + \cos(0)$$

$$= \boxed{\frac{1}{3}x^3 - \cos(x) + 1}$$

$$g'(x) = 3 \cdot \frac{1}{3}x^2 + \sin(x) = \boxed{x^2 + \sin(x).}$$

(c) If we apply the Fundamental Theorem of Calculus to the integral $\int_{-2}^{2} \frac{1}{x^2}\,dx$, we find

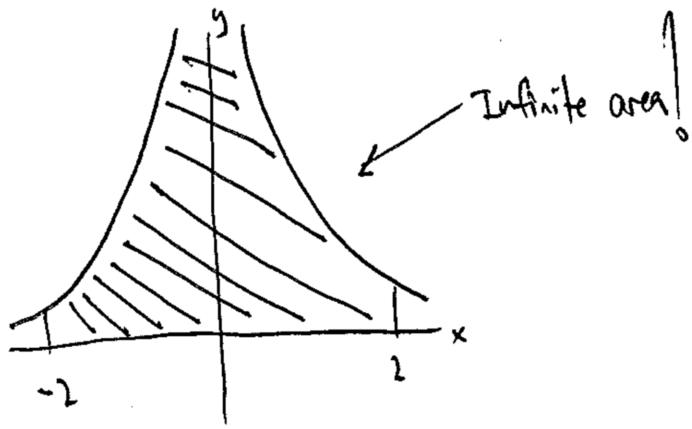
$$\int_{-2}^{2} \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)\Big|_{-2}^{2} = \left(-\frac{1}{2}\right) - \left(-\frac{1}{-2}\right) = -1.$$

But the function $y = 1/x^2$ is always positive, so we expect that the area under the graph is also positive. Where did we go wrong? (Hint: it's not an arithmetic error.)

The function $y = \frac{1}{x^2}$ has a jump discontinuity at $x = 0$. Therefore we cannot apply part 2 of the Fundamental Theorem to evaluate $\int_{-2}^{2} \frac{1}{x^2}\,dx$.

In fact, this integral doesn't exist, or more precisely

$$\int_{-2}^{2} \frac{1}{x^2}\,dx = \infty.$$



Bonus: Add up the first thousand even numbers. What do you get?

$$2+4+6+\dots+1998+2000 = 2002 \cdot 500 = \boxed{1001000}.$$