Integration By Parts Review

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First, what is our equation for doing integration by parts? (You can state it using u, v, du, and dv.)

$$\int u\,dv = uv - \int v\,du$$

Warm-up problems: These problems are straight-forward integration by parts problems.

1. $\int \sqrt{x} \ln(x)\, dx$

$u = \ln(x) \qquad dv = \sqrt{x}\, dx$

$du = \frac{1}{x}\, dx \qquad v = \frac{2}{3} x^{3/2}$

$$\int \sqrt{x} \ln(x) dx = \frac{2}{3} x^{3/2} \ln(x) - \int \frac{2}{3} \frac{x^{3/2}}{x} dx = \frac{2}{3} x^{3/2} \ln(x) - \frac{2}{3} \int x^{1/2} dx$$

$$= \frac{2}{3} x^{3/2} \ln(x) - \frac{4}{9} x^{3/2} + c$$

2. $\int x^2 \cos(x)\, dx$

$u = x^2 \qquad dv = \cos(x) dx$

$du = 2x dx \qquad v = \sin(x) dx$

$$\int x^2 \cos(x) = x^2 \sin(x) - \int 2x \sin(x) dx$$

$$2 \int x \sin(x) dx = 2\left(-x\cos(x) + \int \cos(x) dx\right)$$

$u = x \qquad dv = \sin(x) dx$

$du = dx \qquad v = -\cos(x)$

$$= -2x\cos(x) + 2\sin(x) + c$$

So, $\int x^2 \cos(x) = x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$

The next few problems will be a little more complicated...

3. $\int x^5 e^{x^3}\, dx$

$1^{\underline{st}}$: use substitution

$y = x^3$

$dy = 3x^2 dx$

so $\int x^5 e^{x^3} dx = \frac{1}{3}\int \underbrace{x^3}_{y} \underbrace{e^{x^3}}_{e^y} \underbrace{(3x^2)dx}_{dy}$

$$= \frac{1}{3}\int y e^y dy$$

$2^{\underline{nd}}$: use integration by parts.

$\frac{1}{3}\int y e^y dy \quad = \frac{1}{3}\left(ye^y - \int e^y dy\right)$

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

$$= \frac{1}{3}\left(ye^y - e^y\right) + C$$

$3^{\underline{rd}}$: plug in $y = x^3$.

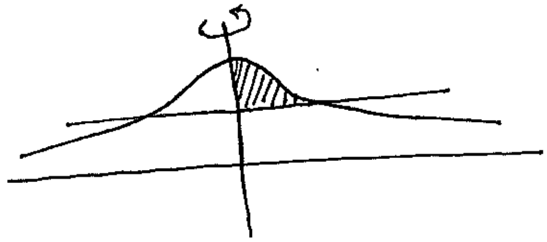
$$\int x^5 e^{x^3} dx = \frac{1}{3}\left(x^3 e^{x^3} - e^{x^3}\right) + C$$

in the first quadrant

4. For this problem, we will find the volume of a solid two ways. Consider the region enclosed by the curves:

$$y = e^{1-x^2} \qquad y = 1 \qquad x = 0$$

(a) Sketch this region.



(b) Find the volume of the solid obtained by rotating this region about the y-axis using disks/washers (slices).

our integral will have dy

So $V = \int_a^b A(y)\,dy$

we have disks, so $A(y) = \pi r^2$

$A(y) = \pi\left(\sqrt{1-\ln(y)}\right)^2 = \pi(1-\ln(y))$

$V = \pi\int_1^e 1 - \ln(y)\,dy = \pi\int_1^e 1\,dy - \pi\int_1^e \ln(y)\,dy$

$= \pi y\Big|_1^e - \pi\int_1^e \ln(y)\,dy$

$= \pi(e-1) - \pi\left(y\ln(y) - y\right)\Big|_1^e$

$= \pi(e-1) - \pi\left((e\ln(e) - e) - (\ln(1) - 1)\right)$

$= \pi(e-1) - \pi(e - e - 0 + 1)$

$= \pi(e-1) - \pi = \pi(e-2)$

Solve for x:

$y = e^{1-x^2}$

$\ln(y) = 1 - x^2$

$x^2 = 1 - \ln(y)$

$x = \sqrt{1-\ln(y)}$

$y = 1$, so $a = 1$

$0 = \sqrt{1-\ln(y)}$

$0 = 1 - \ln(y)$

$\ln(y) = 1$

$y = e$, so $b = e$

$\int \ln(y)\,dy$

$u = \ln(y) \quad dv = dy$

$du = \frac{1}{y}dy \quad v = y$

$= y\ln(y) - \int \frac{y}{y}\,dy$

$= y\ln(y) - \int 1\,dy$

$= y\ln(y) - y + C$

(c) Find the volume of the solid obtained by rotating this region about the y-axis using cylindrical shells.

our integral will have dx.

$$V = \int_a^b 2\pi x h(x)\,dx$$

$$h(x) = e^{1-x^2} - 1$$

$x=0$, so $a=0$

$e^{1-x^2} = 1$

$1-x^2 = \ln(1)$

$1-x^2 = 0$

$x^2 = 1$

$x = \pm 1$, so $b=1$

$$V = \int_0^1 2\pi x\left(e^{1-x^2} - 1\right)dx$$

$$= \underbrace{2\pi \int_0^1 x\,e^{1-x^2}\,dx}\; - 2\pi \int_0^1 x\,dx$$

$$\int_0^1 x e^{1-x^2}\,dx = -\frac{1}{2}\int_0^1 \underbrace{e^{1-x^2}}_{e^u}\underbrace{(-2x)\,dx}_{du} = -\frac{1}{2}\int_1^0 e^u\,du$$

u-substitution!

$u = 1-x^2$

$du = -2x\,dx$

$$= -\frac{1}{2}\left(e^u\right)\Big|_1^0 = -\frac{1}{2}\left(e^0 - e^1\right) = -\frac{1}{2}(1-e)$$

So $$V = 2\pi \int_0^1 x e^{1-x^2}\,dx - 2\pi \int_0^1 x\,dx = 2\pi\left(-\frac{1}{2}(1-e)\right) - 2\pi\left(\frac{x^2}{2}\right)\Big|_0^1$$

$$= 2\pi\left(-\frac{1}{2}(1-e)\right) - 2\pi\left(\frac{1}{2}\right) = \pi(e-1) - \pi = \pi(e-2).$$

Note this is the same answer as part (b)!