1. [9 points] For each question, circle the correct answer.

(a) The area A of a circle of radius 2 is found by:

A) $A = 2\int_{-2}^{2} 2 - x\,dx$

B) $A = 2\pi\int_{-2}^{2} x\sqrt{4-x^2}\,dx$

C) $A = \int_{-2}^{2} \sqrt{4-x^2}\,dx$

D) $A = 4\int_{0}^{2} \sqrt{4-x^2}\,dx$

E) $A = \int_{0}^{2\pi} \sqrt{4-x^2}\,dx$

(b) Evaluate the limit

$$\lim_{x\to 0}\frac{\sin(x)}{x}.$$

A) ∞

B) π

C) 0

D) 1

E) The limit does not exist.

(c) What is the antiderivative of $3\sin(3x)$?

A) $\sin(3x) + C$

B) $\sin^3(x) + C$

C) $9\sin(3x) + C$

D) $\cos(3x) + C$

E) $-\cos(3x) + C$

(d) If $F(x) = \int_0^x e^{-t^2}\, dt$, then what is $F'(x)$?

A) $-2xe^{-x^2}$

B) e^{-x^2}

C) $-\frac{1}{2x}e^{-x^2}$

D) $e^{-x^2} - 1$

E) It's impossible to find it.

(e) Which of the following statements is false?

A) $\int_1^\infty \frac{1}{x}\, dx$ converges.

B) $\int_1^\infty \frac{1}{x^2}\, dx$ converges.

C) $\int_0^1 \frac{1}{x}\, dx$ diverges.

D) $\int_0^1 \frac{1}{x^2}\, dx$ diverges.

E) None of the above.

(f) Consider the region bounded by the x-axis and the curve $y = 2x^2 - x^3$. Rotate this region about the y-axis to form a solid. What method should you use to find its volume, and what variable will appear in your integral?

A) Disks/washers and x.

B) Disks/washers and y.

C) Shells and x.

D) Shells and y.

E) u-substitution and u.

(g) Find the integral

$$\int_0^{2\pi} \sin(x)\,dx.$$

A) 0
B) 1
C) 2
D) π
E) $2\pi^2$

(h) Find the integral

$$\int_0^{\pi/4} \sec^2(x)\,dx.$$

A) 0
B) $\sqrt{2}/2$
C) $\arctan(1)$
D) $1+\ln(\pi/4)$
E) 1

(i) Why are power series useful?

A) They are really neat.
B) They allow your calculator to compute $\ln(2)$ and $e^{48.12}$.
C) They come back with a vengeance in Math 8.
D) They are both theoretically useful and have wide applications to physics.
E) All of the above.

2. [4 points] Find $\int \cos^3(x)\,dx$.

$$\int \cos^3 x\,dx = \int \cos x\,(1-\sin^2 x)\,dx. \qquad \text{let } u=\sin x, \text{ so } du=\cos x\,dx$$

$$= \int 1-u^2\,du$$

$$= u - \frac{u^3}{3} + C$$

$$= \boxed{\sin x - \frac{\sin^3 x}{3} + C}$$

3. [4 points] Find $\int x\ln(x)\,dx$.

IBP. $\quad u = \ln(x) \qquad v = \dfrac{x^2}{2}$

$\quad du = \dfrac{1}{x}\,dx \qquad dv = x\,dx$

$$\int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \int \frac{1}{x}\,\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx$$

$$= \boxed{\frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C_6}$$

4. [5 points] Define the integral $\int_a^b f(x)\,dx$ in terms of Riemann sums.

Break up the interval $[a,b]$ into subintervals $[x_0,x_1]$, $[x_1,x_2]$, ..., $[x_{n-1},x_n]$ of equal width $\Delta x = \frac{b-a}{n}$. Let x_1^* be any point in $[x_0,x_1]$, x_2^* any point in $[x_1,x_2]$, ..., and x_n^* be any point in $[x_{n-1},x_n]$. Define

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x\right),$$

provided this limit exists and is independent of choice of sample points.

5. [6 points] Find $\int_{\pi/4}^{\pi/2} 2\cot x\,dx$. (For a bonus point, simplify your answer as much as possible.)

$$\int_{\pi/4}^{\pi/2} 2\cot x\,dx = 2\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx.$$

Let $u = \sin x$, so $du = \cos x\,dx$.

$x = \frac{\pi}{2} \rightsquigarrow u = 1$

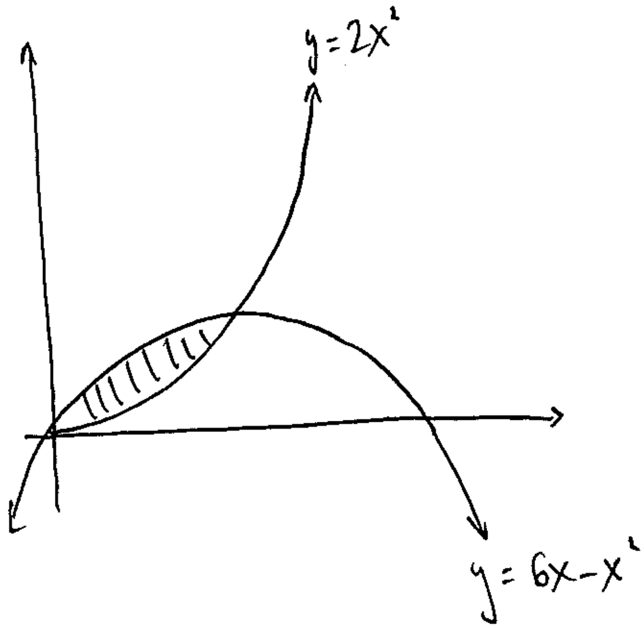
$x = \frac{\pi}{4} \rightsquigarrow u = \frac{1}{\sqrt{2}}$

$$= 2\int_{1/\sqrt{2}}^{1} \frac{1}{u}\,du$$

$$= 2\ln(u)\Big|_{1/\sqrt{2}}^{1} = -2\ln\left(\frac{1}{\sqrt{2}}\right) = 2\ln(\sqrt{2}) = \boxed{\ln(2)}$$

6. [6 points] Find the area between the two curves

$$y = 6x - x^2 \qquad\qquad y = 2x^2.$$



Points of Intersection

$2x^2 = 6x - x^2$

$3x^2 = 6x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, 2$

$\text{Area} = \int_0^2 6x - x^2 - 2x^2 \, dx$

$= \int_0^2 6x - 3x^2 \, dx$

$= 3x^2 - x^3 \Big|_0^2$

$= 3 \cdot 4 - 8 = \boxed{4}$

7. [8 points] **Circle ONE of the following questions and answer it.**

(a) A car brakes with a constant deceleration of 10 ft/s^2. It skids 500 feet before coming to a stop. How fast was the car traveling when the brakes were applied?

(b) You want to fence in a rectangular field with an area of 1000 square feet. However, three of the sides have to be reinforced. The fence on the reinforced sides cost 4 times as much as the fence on the non-reinforced side. What should the dimensions of this rectangular field be in order to minimize the cost of this project?

a) $a(t) = -10$

$v(t) = -10t + C$

$s(t) = -5t^2 + Ct$

We wish to find C. Let t_1 be the time at which the car stops moving. Then

$0 = v(t_1) = -10t_1 + C \Rightarrow t_1 = \frac{C}{10}$.

We have

$$500 = s(t_1) = -5t_1^2 + Ct_1 = -5\left(\frac{C}{10}\right)^2 + C\left(\frac{C}{10}\right)$$

$$= -\frac{C^2}{20} + \frac{C^2}{10} = \frac{C^2}{20}$$

Thus $C^2 = 10000$, and $C = 100$. $\boxed{100 \text{ ft/s}}$

b) $\text{Cost} = C = 5x + 8y$. $xy = 1000 \Rightarrow y = \frac{1000}{x}$.

$C(x) = 5x + \frac{8000}{x}$

$C'(x) = 5 - \frac{8000}{x^2}$.

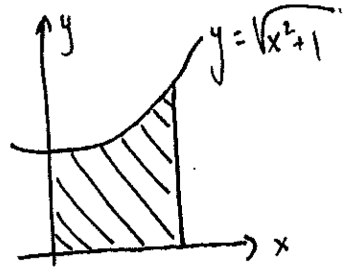
$C'(x) = 0 \Rightarrow x^2 = 1600$

$\Rightarrow x = 40$

$\Rightarrow y = 25$

$\boxed{25 \times 40 \text{ feet}}$

8. [12 points] Consider the region below the curve $y = \sqrt{x^2+1}$ from $x = 0$ to $x = 1$ as pictured below.



(a) Rotate this region about the x-axis to form a solid. What is its volume?

Disks, dx:

$$V = \int_0^1 \pi \left(\sqrt{x^2+1}\right)^2 dx$$

$$= \pi \int_0^1 x^2+1 \, dx$$

$$= \pi \left. \frac{x^3}{3} + x \right|_0^1 = \boxed{\frac{4\pi}{3}}$$

(b) Now rotate this region about the y-axis to form a different solid. What is its volume?

Shells, dx:

$$V = \int_0^1 2\pi x \sqrt{x^2+1} \, dx$$

$$= \pi \int_0^1 2x\sqrt{x^2+1} \, dx \qquad \left[\text{u-sub: } u = x^2+1\right]$$

$$= \pi \frac{2}{3}\left(x^2+1\right)^{3/2} \Big|_0^1$$

$$= \boxed{\pi \frac{2}{3}\left(2^{3/2} - 1\right)}$$

Disks, dy: Break up region into a square and a washer region.

$$V = \pi + \pi \int_1^{\sqrt{2}} 1 - (y^2 - 1) \, dy$$

$$= \pi + \pi \int_1^{\sqrt{2}} 2 - y^2 \, dy$$

$$= \pi + \pi \left[2y - \frac{y^3}{3} \right]_1^{\sqrt{2}}$$

$$= \pi \left[1 + 2\sqrt{2} - \frac{2\sqrt{2}}{3} - 2 + \frac{1}{3} \right]$$

these numbers are indeed the same

9. [8 points] The following question involves the Fundamental Theorem of Calculus.

(a) State the Fundamental Theorem of Calculus (both parts).

Part 1: Suppose $f(t)$ is continuous on $[a,b]$. Define $g(x) = \int_a^x f(t)\,dt$.

Then $g(x)$ is differentiable and $g'(x) = f(x)$,

Part 2: Suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative.

Then $\int_a^b f(x)\,dx = F(x)\Big|_a^b$.

(b) Why is the Fundamental Theorem of Calculus the most important theorem we've learned this quarter? Give two reasons.

- FTOC is the bridge between differential calculus (math 1) and integral calculus (math 2)
- FTOC part 2 allows us to integrate if we know how to antidifferentiate
- FTOC part 1 gives the existence of antiderivatives

10. [10 points] Consider the indefinite integral

$$\int \frac{x}{\sqrt{x^2+1}}\,dx.$$

(a) Find this integral using a u-substitution.

Let $u = x^2+1$. Then $du = 2x\,dx$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C$$

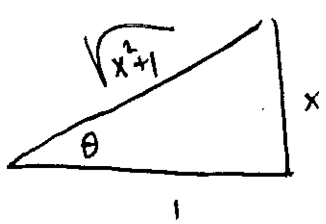
$$= \boxed{\sqrt{x^2+1} + C}$$

(b) Find this integral using a trig(onometric) substitution.

Let $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{\tan\theta}{\sec\theta}\sec^2\theta\,d\theta = \int \tan\theta\sec\theta\,d\theta = \sec\theta + C$$

$$= \boxed{\sqrt{x^2+1} + C}$$



11. [9 points] Compute the integral

$$\int_0^\infty e^{-\sqrt{x}}\,dx.$$

$$= \lim_{t\to\infty} \int_0^t e^{-\sqrt{x}}\,dx.$$

Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$

So $dx = 2u\,du$

$$= \lim_{t\to\infty} \int_0^{\sqrt{t}} e^{-u}\, 2u\,du = \lim_{t\to\infty} 2\int_0^{\sqrt{t}} u e^{-u}\,du. = *$$

Using IBP, $\int u e^{-u}\,du = -ue^{-u} + \int e^{-u}\,du$

$$= -ue^{-u} - e^{-u} = e^{-u}(-u-1).$$

$$* = \lim_{t\to\infty} 2\left[e^{-u}(-u-1) \right]_0^{\sqrt{t}}$$

$$= \lim_{t\to\infty} \underbrace{2e^{-\sqrt{t}}(-\sqrt{t}-1)}_{\to 0\text{, by l'Hôpital}} + 2 = \boxed{2.}$$

12. [4 points] Use trig(onometric) substitution to prove

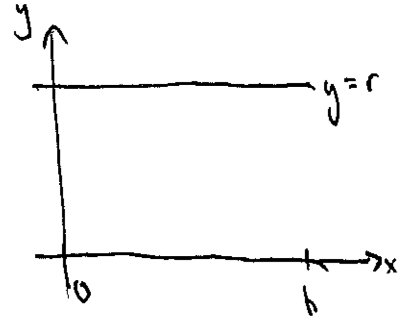
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C.$$

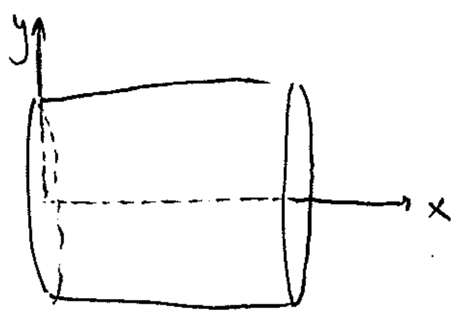
Let $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\cos\theta}\cos\theta\, d\theta = \int d\theta = \theta + C$$

$$= \arcsin(x) + C.$$

13. [7 points] Using solids of revolution and the disk/washer method, show that the formula for the volume of a cylinder with height h and radius r is $V = \pi r^2 h$.





Consider the line $y=r$. Rotate this about the x-axis to form a solid of revolution. It will form a cylinder. If we consider the line segment from $x=0$ to $x=h$, the resulting cylinder will have height h.

Using disks,

$$V = \int_0^h \pi r^2\,dx = \pi r^2 x \Big|_0^h = \pi r^2 h.$$

14. [4 points] Write 2 or 3 sentences about any extra topic we mentioned after finishing improper integrals.

15. [4 points] Explain two reasons why we feel justified in taking points off of your exam when you forget to write dx.

- The dx indicates the variable of integration
- When you make a u-sub or trig-sub, the dx changes to du ~~[illegible]~~ or $d\theta$ in a nice way
- The dx is ~~[illegible]~~ reminiscent of the Δx in Riemann sums
- The dx indicates a differential.