Math 2 Due: January 11, 2008

Sample Written Problem 1

Solution

It is the end of fall term. You and your classmates are very busy and stressed over finals. Your Math 1 teacher has passed out a “study guide” for the final. This study guide does not contain any practice problems, solutions, definitions, page numbers, sections, notes or anything that seems helpful. The “study guide” is just a long list of words. Because you and your friends are so busy, you each decide to take one of the terms overnight and report back your findings the next day. The term you have is LIMITS AT INFINITY. The next day comes but there is no time to meet. You must write up everything you’ve learned and slip a copy under each of your friends’ doors. Include **all** relevant definitions, a possible test problem and solution, and a little bit about why it is important. Remember how unhelpful the long list of terms was, make your write up readable and helpful, like a story.

A limit of a function at infinity looks like this:

$$\lim_{x\to\infty} f(x) = L.$$

For a function f defined on an interval (a,∞), $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. Intuitively $\lim_{x\to\infty} f(x) =?$ is asking, “What happens to the function $f(x)$ as I plug in larger and larger values of x?” If it gets close to one number, L, then that is the limit of the function at infinity.

Even more accurately, for a function $f(x)$, defined on an interval (a,∞), we define

$$\lim_{x\to\infty} f(x)$$

to mean that for every $\epsilon > 0$ there is a corresponding number N such that

$$\text{if } x > N \text{ then } |f(x) - L| < \epsilon.$$

The exact same ideas exist for limits at negative infinity. so, $\lim_{x\to-\infty} f(x)$ means look at the value of $f(x)$ as we plug in larger and larger negative numbers.

Sometimes this limit does not exist. For example,

$$\lim_{x\to\infty} \cos x.$$

The cosine function oscillates between -1 and 1 even as we plug in larger and larger numbers. It never ”settles” on where it wants to be. We say this limit does not exist.

Sometimes a function just keeps growing larger and larger with larger and larger values of x. Like $f(x) = x^2$. We would say,

$$\lim_{x\to\infty} x^2 = \infty$$

Formally, $\lim_{x\to\infty} f(x) = \infty$ means that for every positive number M there is a corresponding positive number N such that

$$\text{if } x > N \text{ then } f(x) > M.$$

A typical problem may look like this:

$$\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}.$$

A good method to solve these kinds of problems is dividing the top and bottom of the fraction by the highest power of x, and remembering that $\lim_{x\to infty} \frac{1}{x^r} = 0$ if r is a rational number greater than zero. The solution would look like this:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} &= \lim_{x\to\infty} \frac{\frac{\sqrt{9x^6 - x}}{x^3}}{\frac{x^3+1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{9x^6 - x}{x^6}}}{\frac{x^3}{x^3} + \frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{9x^6}{x^6} - \frac{x}{x^6}}}{1 + \frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\sqrt{9 - \frac{1}{x^5}}}{1 + \frac{1}{x^3}}
\end{aligned}$$

Now, we can think about each piece individually. The constants don't change as x goes to infinity, and both $\frac{1}{x^5}$ and $\frac{1}{x^3}$ go to zero. So we get $\frac{\sqrt{9}}{1}$ which is 3. YAY!

The reason limits at infinity are useful is horizontal asymptotes. If $\lim_{x\to\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote for the graph of $f(x)$.