Math 2 Name: Solutions
January 25, 2008

Quiz 3

Show your work, and write clearly. No textbooks, notes, or calculators.

1. (3 points) Find $\int_0^5 x\,dx$ in three ways:
   (a) Use the definition of the definite integral as the limit of a Riemann Sum.
   (b) Interpret it as an area, which can be found using a familiar geometry formula.
   (c) Use the Fundamental Theorem of Calculus, which tells us that $\int_0^5 x\,dx = F(5) - F(0)$, where $F(x)$ is any antiderivative of x on $[0,5]$.

Problem 1 Hint: For part (a), use right endpoints, and then the formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Obviously, your answers to (a), (b), and (c) should all be the same number!

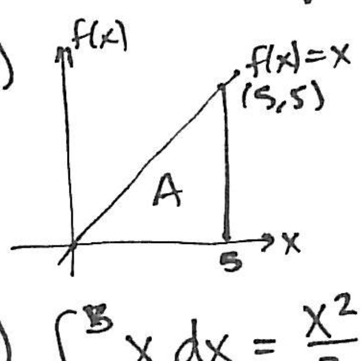
a) $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ $\quad \Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$

$\Delta x = \frac{5}{n} \quad x_i = \frac{5i}{n}$

$\int_0^5 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5i}{n}\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{25i}{n^2} = \lim_{n\to\infty} \frac{25}{n^2} \sum_{i=1}^{n} i$

$= \lim_{n\to\infty} \frac{25}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{25}{2}$ ✓

b) 

$A = \frac{1}{2}bh = \frac{1}{2}(5)(5) = \frac{25}{2}$ ✓

c) $\int_0^5 x\,dx = \frac{x^2}{2}\Big|_0^5 = \frac{25}{2} - 0 = \frac{25}{2}$ ✓

$f(x) = x$

$F(x) = \frac{x^2}{2}$

2. (5 points) Let f and g be continuous functions on an interval $[a, b]$. For each of the following statements, say if it is **always** true, or not. If it is not always true, give an example of why, two functions, f and g, that make the statement false.

(a) $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(b) $\int_a^b f(x) - g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(c) $\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$

(d) $\int_a^b \frac{f(x)}{g(x)}dx = \frac{\int_a^b f(x)\,dx}{\int_a^b g(x)\,dx}$, provided all denominators are nonzero.

(e) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

Problem 2 Hint: Each false statement can be shown to be false by considering the functions f(x) = x and g(x)=1 on [0,5].

a) TRUE pg 373

b) TRUE pg 373

c) FALSE $a=0, b=5, f=x, g=1$

Problem #1

$$\int_0^5 (x)(1)dx = \int_0^5 x\,dx = \frac{25}{2}$$

$$\left(\int_0^5 x\,dx\right)\left(\int_0^5 1\,dx\right) = \left(\frac{25}{2}\right)\left[1(5-0)\right] = \frac{125}{2}$$

not =

d) FALSE $a=0, b=5, f(x)=x, g(x)=1$

$$\int_0^5 \frac{x}{1}dx = \int_0^5 x\,dx = \frac{25}{2}$$

$$\frac{\int_0^5 x\,dx}{\int_0^5 1\,dx} = \frac{\frac{25}{2}}{1(5-0)} = \frac{25}{2\cdot 5} = \frac{5}{2}$$

not =

e) TRUE pg 373

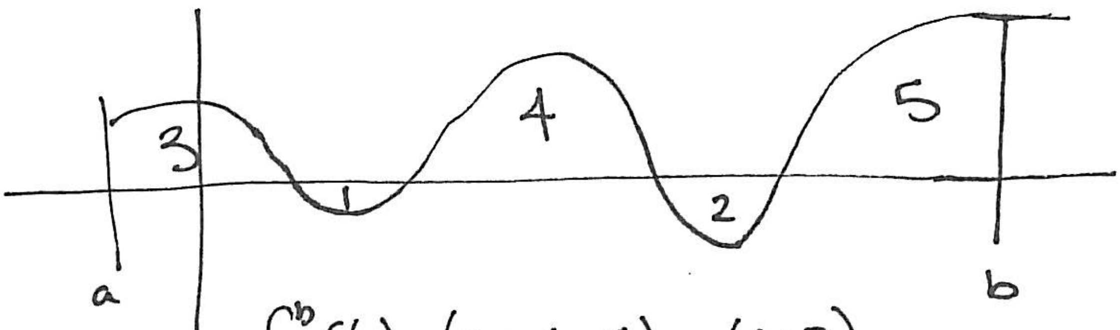
3. (2 points) Recall that in general, $\int_a^b f(x)dx$ represents the net area under the graph of f from a to b.

(a) Suppose the region bounded between f and the x-axis between $x = a$ and $x = b$ has five parts: three parts above the x-axis (i.e. where f is positive) with areas 3, 4, and 5, and two parts below the x-axis (i.e. where f is negative) with areas 1 and 2. Find $\int_a^b f(x)dx$.

(b) Find $\int_0^{3\pi} \sin x \, dx$, given that $\int_0^{\pi} \sin x \, dx = 2$.

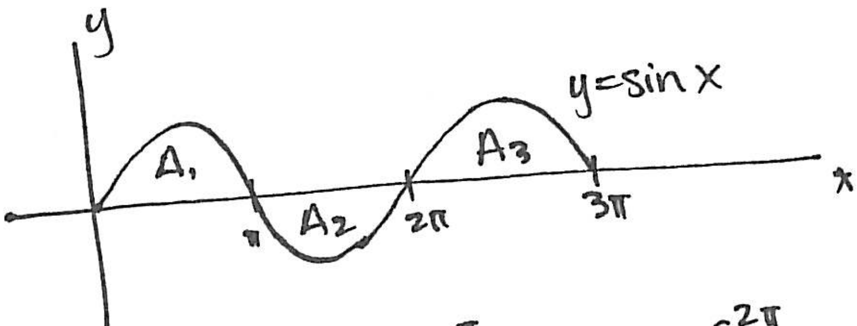
Problem 3 Hint: Try drawing pictures. In part (a), the answer is not just 3+4+5+1+2. For part (b), remember that the graph of $y = \sin x$ has the same shape on the intervals $[0, \pi]$, $[\pi, 2\pi]$ and $[2\pi, 3\pi]$ except its flipped upside-down on the middle interval.

a) f could look like:



$$\int_a^b f(x) = (3+4+5) - (1+2)$$
$$= 12-3=9$$

b)



$$A_1 = A_2 = A_3$$
$$= \int_0^{\pi} \sin x dx$$
$$= 2$$

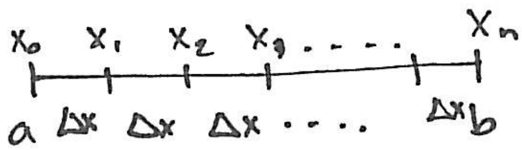
$$\int_0^{3\pi} \sin x \, dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} \sin x dx + \int_{2\pi}^{3\pi} \sin x dx$$
$$= A_1 - A_2 + A_3 = 2-2+2 = 2.$$

BONUS (1 point) Explain where the formula you used in 1(a) comes from. Be sure to include what each piece of the formula refers to, how to calculate it, and how it all fits together in a picture.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \Delta x = \frac{b-a}{n}$$

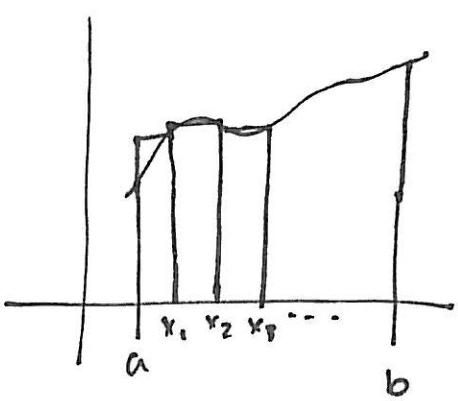
$$x_i = a + i\Delta x$$

We split up the interval $[a,b]$ into 'n' subintervals of equal length, Δx.



So Δx is the length of the interval, $b-a$, divided by the number of subintervals, $n \Rightarrow \Delta x = \frac{b-a}{n}$.

x_i is the right edge of each interval, ~~wie~~ which is "i" Δx units away ~~for~~ from a, $x_i = a + i\Delta x$. Now we look at the height of the function, f, at each x_i. $f(x_i)$ gives the height of a rectangle, the i^{th} rectangle used to approximate area. The width of the rectangle is Δx, so the area of each is $f(x_i)\Delta x$.



We add all these up: $\sum_{i=1}^{n} f(x_i)\Delta x$. We know our actual area, $\int_a^b f(x)dx$, is given by better and better approximations, with more rectangles. So we take the limit: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.