MATH 2 SOLUTIONS TO PROBLEM SET #17

SECTION 7.4 - PARTIAL FRACTIONS

(9.) $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

$$x-9 = A(x-2) + B(x+5)$$

$$x-9 = (A+B)x + (-2A+5B)$$

$$\begin{cases} A+B=1 \\ -2A+5B=-9 \end{cases}$$

FIRST EQUATION GIVES $B = 1-A$. PLUGGING THIS INTO THE 2^{ND} EQUATION, WE HAVE $-2A + 5(1-A) = -9$ SO $-7A + 5 = -9 \Rightarrow -7A = -14$ $\Rightarrow A = 2$. THUS $B = 1 - A = -1$.

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \left(\frac{2}{x+5} - \frac{1}{x-2}\right) dx$$

$$= \boxed{2\ln|x+5| - \ln|x-2| + C.}$$

(CHECK BY DIFFERENTIATION, AS ALWAYS.)

$\boxed{10.}$ $\displaystyle\int \frac{1}{(t+4)(t-1)}\,dt$

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1}$$

$$1 = (t-1)A + (t+4)B$$

$$1 = (A+B)t + (-A+4B)$$

$$\begin{cases} A+B=0 \\ -A+4B=1 \end{cases}$$

FIRST EQUATION GIVES $B=-A$. PLUGGING THIS INTO THE 2^{ND} EQUATION, WE HAVE $-5A=1$

SO $A=-\frac{1}{5}$, $B=\frac{1}{5}$.

$$\frac{1}{(t+4)(t-1)} = \frac{1}{5}\left(\frac{-1}{t+4} + \frac{1}{t-1}\right).$$

$$\int \frac{1}{(t+4)(t-1)}\,dt = \int \frac{1}{5}\left(\frac{-1}{t+4} + \frac{1}{t-1}\right)dt$$

$$= \frac{1}{5}\int\left(\frac{1}{t-1} - \frac{1}{t+4}\right)dt = \boxed{\frac{1}{5}\left(\ln|t-1| - \ln|t+4|\right) + C}.$$

(11.) $\int_2^3 \frac{1}{x^2-1}\, dx$

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$1 = A(x-1) + B(x+1)$$
$$1 = (A+B)x + (B-A)$$

$$\begin{cases} A+B=0 \\ B-A=1 \end{cases}$$

FIRST EQUATION GIVES $B=-A$. PLUGGING THIS INTO THE 2^{ND} EQUATION, WE HAVE $-2A=1$, THUS $A=-\frac{1}{2}$, $B=\frac{1}{2}$.

$$\frac{1}{x^2-1} = \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$\int_2^3 \frac{1}{x^2-1}\, dx = \int_2^3 \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \frac{1}{2}\int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \frac{1}{2}\Big[\ln|x-1| - \ln|x+1|\Big]_2^3 = \frac{1}{2}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_2^3$$

$$= \frac{1}{2}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{3}\right)\right) = \boxed{\frac{1}{2}\ln\left(\frac{3}{2}\right)}.$$

(14.) $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

$$1 = A(x+b) + B(x+a)$$

$$1 = (A+B)x + (bA + aB)$$

$$\begin{cases} A + B = 0 \\ bA + aB = 1 \end{cases}$$

FIRST EQUATION GIVES $B = -A$.
PLUGGING THIS INTO THE 2ND EQUATION,
$(b-a)A = 1$, SO $A = \dfrac{1}{b-a}$ (PROVIDED $b \neq a$)

AND THUS $B = -\dfrac{1}{b-a}$, SO

$$\frac{1}{(x+a)(x+b)} = \frac{\frac{1}{b-a}}{x+a} - \frac{\frac{1}{b-a}}{x+b} = \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right),$$

FOR $a \neq b$.

THUS FOR $a \neq b$,

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx$$

$$= \frac{1}{b-a}\int\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C$$

$$= \boxed{\frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C \quad (b \neq a)}.$$

ON THE OTHER HAND, IF $b = a$, THEN $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{(x+a)^2}$.

$$\int \frac{1}{(x+a)^2}\,dx = \int (x+a)^{-2}\,dx \underset{\boxed{u = x+a}}{=} \boxed{-\frac{1}{x+a} + C \quad (b=a)}.$$

NOTE: OKAY FOR CREDIT IF THEY JUST SHOW THE CASE FOR $a \neq b$.

(57.) (a.) LET $t = \tan\left(\frac{x}{2}\right)$, $-\pi < x < \pi$.

THUS $t = \dfrac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$.

NOW, $\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) = 1$,

SO $\sin\left(\frac{x}{2}\right) = \pm\sqrt{1-\cos^2\left(\frac{x}{2}\right)}$ $\left(- \text{ IF } -\pi < x < 0,\ + \text{ IF } 0 \le x < \pi.\right)$

THUS $t = \dfrac{\pm\sqrt{1-\cos^2\left(\frac{x}{2}\right)}}{\cos\left(\frac{x}{2}\right)}$

$$\Rightarrow t^2 = \frac{1-\cos^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} \Rightarrow t^2 = \frac{1}{\cos^2\left(\frac{x}{2}\right)} - 1$$

$$\Rightarrow t^2 + 1 = \frac{1}{\cos^2\left(\frac{x}{2}\right)} \Rightarrow \cos^2\left(\frac{x}{2}\right) = \frac{1}{t^2+1}$$

$$\Rightarrow \cos\left(\frac{x}{2}\right) = \frac{1}{\pm\sqrt{t^2+1}}$$

$$\Rightarrow \boxed{\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}}}$$ (SINCE $\cos\left(\frac{x}{2}\right) > 0$ FOR $-\pi < x < \pi$).

SIMILARLY, $\cos\left(\frac{x}{2}\right) = \sqrt{1-\sin^2\left(\frac{x}{2}\right)}$

SO $t = \dfrac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \dfrac{\sin\left(\frac{x}{2}\right)}{\sqrt{1-\sin^2\left(\frac{x}{2}\right)}}$

$$\Rightarrow t^2 = \frac{\sin^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)} \Rightarrow \frac{1-\sin^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)} = \frac{1}{t^2}$$

(UNLESS $x = 0$, IN WHICH CASE WE CLEARLY HAVE $\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$.)

$$\Rightarrow \frac{1}{\sin^2\left(\frac{x}{2}\right)} - 1 = \frac{1}{t^2} \Rightarrow \frac{1}{\sin^2\left(\frac{x}{2}\right)} = \frac{1}{t^2} + 1 = \frac{1+t^2}{t^2}$$

$$\Rightarrow \sin^2\left(\frac{x}{2}\right) = \frac{t^2}{1+t^2} \Rightarrow \boxed{\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}}$$

(SINCE $t > 0 \Leftrightarrow \sin\left(\frac{x}{2}\right) > 0$.)

(b.) FOLLOWS IMMEDIATELY FROM THE FORMULAS FOR SINE AND COSINE OF A DOUBLE ANGLE:

$$\sin(2a) = 2\sin(a)\cos(a)$$
$$\cos(2a) = \cos^2(a) - \sin^2(a).$$

(USE $a = \frac{x}{2}$.)

(c.) $t = \tan\left(\frac{x}{2}\right)$

$\Rightarrow \frac{x}{2} = \arctan t \quad \Rightarrow \quad x = 2\arctan t$

$\Rightarrow \frac{dx}{dt} = \frac{d}{dt} 2\arctan t = 2 \frac{d}{dt} \arctan t$

$= \frac{2}{1+t^2}$. THUS $dx = \frac{2}{1+t^2} dt$.

(59.) $\displaystyle\int \frac{1}{3\sin x - 4\cos x}\, dx$

$$\boxed{\begin{array}{l} \text{LET } \quad t = \tan\left(\frac{x}{2}\right) \\ \sin x = \dfrac{2t}{1+t^2} \\ \cos x = \dfrac{1-t^2}{1+t^2} \\ dx = \dfrac{2}{1+t^2}\, dt \end{array}}$$

$$\int \frac{1}{3\sin x - 4\cos x}\, dx = \int \frac{1}{\frac{6t}{1+t^2} - \frac{4(1-t^2)}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1}{3t - 2(1-t^2)}\, dt = \int \frac{1}{2t^2+3t-2}\, dt$$

$$= \frac{1}{2}\int \frac{1}{t^2+\frac{3}{2}t-1}\, dt = \frac{1}{2}\int \frac{1}{\left(t+\frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2}\, dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)(t+2)}\, dt = \frac{1}{5}\int\left(\frac{1}{t-\frac{1}{2}} - \frac{1}{t+2}\right) dt$$

$$\boxed{\begin{array}{l} \dfrac{1}{\left(t-\frac{1}{2}\right)(t+2)} = \dfrac{A}{t-\frac{1}{2}} + \dfrac{B}{t+2} \\ 1 = A(t+2) + B\left(t-\frac{1}{2}\right) \\ 1 = (A+B)t + \left(2A - \frac{1}{2}B\right) \\ \begin{cases} A+B = 0 \\ 2A - \frac{1}{2}B = 1 \end{cases} \\ \text{so } B = -A,\ A = \frac{2}{5},\ B = -\frac{2}{5} \\ \dfrac{1}{\left(t-\frac{1}{2}\right)(t+2)} = \dfrac{2}{5}\left(\dfrac{1}{t-\frac{1}{2}} - \dfrac{1}{t+2}\right) \end{array}}$$

$$= \frac{1}{5}\left(\ln\left|t-\tfrac{1}{2}\right| - \ln|t+2|\right) + C$$

$$= \frac{1}{5}\ln\left|\frac{t-\frac{1}{2}}{t+2}\right| + C$$

$$= \boxed{\frac{1}{5}\ln\left|\frac{\tan\left(\frac{x}{2}\right)-\frac{1}{2}}{\tan\left(\frac{x}{2}\right)+2}\right| + C}$$

(NOTE: AGREES WITH BACK OF BOOK, SINCE DIFFERS BY $\ln 2$, A CONSTANT. ADJUST THE ARBITRARY C.)