# MATH 2 SOLUTIONS TO PROBLEM SET # 16

## SECTION 7.3: TRIGONOMETRIC SUBSTITUTION

(1.) $$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{27\sec^2\theta\tan\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$\boxed{\begin{aligned} x &= 3\sec\theta \\ x^2 &= 9\sec^2\theta \\ \sqrt{x^2-9} &= \sqrt{9(\sec^2\theta-1)} = 3\tan\theta \\ dx &= 3\sec\theta\tan\theta\,d\theta \end{aligned}}$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$$

$$\boxed{\begin{aligned} \frac{x}{3} &= \sec\theta \\ \cos\theta &= \frac{3}{x} \\ \sin\theta &= \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{x^2}} \end{aligned}}$$

$$= \frac{1}{9}\sqrt{1-\frac{9}{x^2}} + C = \frac{1}{9}\sqrt{\frac{x^2-9}{x^2}} + C = \boxed{\frac{\sqrt{x^2-9}}{9x} + C}$$

CHECK: $$\frac{d}{dx}\frac{\sqrt{x^2-9}}{9x} = \frac{(9x)\cdot\frac{1}{2}(x^2-9)^{-\frac{1}{2}}\cdot 2x - \sqrt{x^2-9}\cdot 9}{81x^2}$$

$$= \frac{9x^2-9(x^2-9)}{81x^2\sqrt{x^2-9}} = \frac{x^2-(x^2-9)}{9x^2\sqrt{x^2-9}} = \frac{1}{x^2\sqrt{x^2-9}} \quad ✓$$

$\boxed{(2.)}$ $\int x^3\sqrt{9-x^2}\,dx = \int 81\sin^3\theta\cos\theta \cdot 3\cos\theta\,d\theta$

$$\boxed{\begin{aligned} x &= 3\sin\theta \\ x^3 &= 27\sin^3\theta \\ \sqrt{9-x^2} &= \sqrt{9(1-\sin^2\theta)} = 3\cos\theta \\ dx &= 3\cos\theta\,d\theta \end{aligned}}$$

$= 243\int \sin^3\theta\cos^2\theta\,d\theta = 243\int \sin^2\theta\cos^2\theta \cdot \sin\theta\,d\theta$

$= 243\int (1-\cos^2\theta)\cos^2\theta \cdot \sin\theta\,d\theta$

$= 243\int (\cos^2\theta - 1)\cos^2\theta \cdot -\sin\theta\,d\theta$

$= 243\int \cos^4\theta - \cos^2\theta \cdot -\sin\theta\,d\theta$

$$\boxed{\begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\,d\theta \end{aligned}}$$

$= 243\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$

$= 243\left(\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right) + C$

$$\boxed{\begin{aligned} \sin\theta &= \frac{x}{3} \\ \cos\theta &= \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x}{3}\right)^2} = \sqrt{1-\frac{x^2}{9}} = \frac{(9-x^2)^{1/2}}{3} \end{aligned}}$$

$= 243\left(\frac{(9-x^2)^{5/2}}{5\cdot 3^5} - \frac{(9-x^2)^{3/2}}{3\cdot 3^3}\right) + C$

$= \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C = \boxed{(9-x^2)^{3/2}\left(\frac{-(x^2+6)}{5}\right) + C}$

CHECK BY DIFFERENTIATION. ✓

(5.) $$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\, dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^3\theta \cdot \tan\theta} \cdot \sec\theta \tan\theta \, d\theta$$

$t = \sec\theta$

$t^3 = \sec^3\theta$

$\sqrt{t^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$dt = \sec\theta \tan\theta \, d\theta$

$\cos\theta = \frac{1}{t}$

$t = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$t = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta}\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta \, d\theta$$

$$= \left[\frac{\theta + \sin\theta\cos\theta}{2}\right]_{\frac{\pi}{4}}^{\pi/3}$$

$$= \left[\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right) - \left(\frac{\pi}{8} + \frac{1}{4}\right)\right]$$

$$= \boxed{\frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}}.$$

$\boxed{(6.)}$ $$\int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta} \cdot \sec\theta \tan\theta \, d\theta$$

$x = \sec\theta$

$\sqrt{x^2-1} = \tan\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$\cos\theta = \frac{1}{x}$

$x = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$x = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

$$= \int_0^{\pi/3} \tan^2\theta \, d\theta = \int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta$$

$$= \left[\tan\theta - \theta\right]_0^{\pi/3}$$

$$= \left[\left(\sqrt{3} - \frac{\pi}{3}\right) - (0 - 0)\right]$$

$$= \boxed{\sqrt{3} - \frac{\pi}{3}}.$$

(9.) $\displaystyle\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{1}{4\sec\theta}\cdot 4\sec^2\theta\, d\theta = \int \sec\theta\, d\theta$

$$\boxed{\begin{aligned} x &= 4\tan\theta \\ \sqrt{x^2+16} &= 4\sec\theta \\ dx &= 4\sec^2\theta\, d\theta \end{aligned}}$$

$= \ln|\sec\theta + \tan\theta| + c$

$= \ln\left|\dfrac{\sqrt{x^2+16}+x}{4}\right| + c$

$= \ln\left(\dfrac{\sqrt{x^2+16}+x}{4}\right) + c = \ln(\sqrt{x^2+16}+x) \underbrace{- \ln 4 + c}_{\text{NEW } c}$

$= \boxed{\ln(\sqrt{x^2+16}+x) + c}$.

CHECK: $\dfrac{d}{dx}\ln(\sqrt{x^2+16}+x) = \dfrac{1}{\sqrt{x^2+16}+x}\cdot\left(\tfrac{1}{2}(x^2+16)^{-\frac{1}{2}}\cdot 2x + 1\right)$

$= \dfrac{1}{\sqrt{x^2+16}+x}\left(1 + \dfrac{x}{\sqrt{x^2+16}}\right) = \dfrac{1}{\sqrt{x^2+16}+x}\left(\dfrac{\sqrt{x^2+16}+x}{\sqrt{x^2+16}}\right)$

$= \dfrac{1}{\sqrt{x^2+16}}$. ✓

(39.) (a.) $\displaystyle\int_0^x \sqrt{a^2-t^2}\, dt = \int_0^{\sin^{-1}(\frac{x}{a})} a^2\cos^2\theta\, d\theta$

$$\boxed{\begin{aligned} t &= a\sin\theta \\ \sqrt{a^2-t^2} &= a\cos\theta \\ dt &= a\cos\theta\, d\theta \end{aligned}}$$

$\displaystyle = a^2\int_0^{\sin^{-1}(\frac{x}{a})} \cos^2\theta\, d\theta$

$$\boxed{\begin{aligned} &\theta = \sin^{-1}\left(\tfrac{t}{a}\right) \\ &t = 0 \Rightarrow \theta = 0 \\ &t = x \Rightarrow \theta = \sin^{-1}\left(\tfrac{x}{a}\right) \end{aligned}}$$

$= a^2\left[\dfrac{\theta + \sin\theta\cos\theta}{2}\right]_0^{\sin^{-1}(\frac{x}{a})}$

$= a^2\left(\tfrac{1}{2}\sin^{-1}\left(\tfrac{x}{a}\right) + \tfrac{1}{2}\left(\tfrac{x}{a}\right)\sqrt{1-\left(\tfrac{x}{a}\right)^2}\right)$

$= \boxed{\tfrac{1}{2}a^2\sin^{-1}\left(\tfrac{x}{a}\right) + \tfrac{1}{2}x\sqrt{a^2-x^2}}$.

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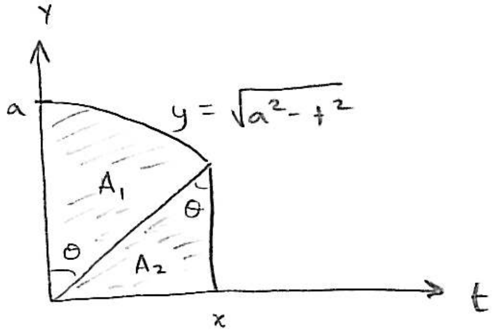
(b.)




$$\int_0^x \sqrt{a^2 - t^2}\, dt = A_1 + A_2 = \frac{\theta}{2\pi} \cdot \pi a^2 + \frac{1}{2} x \sqrt{a^2 - x^2}$$

$$= \frac{\theta}{2} a^2 + \frac{1}{2} x \sqrt{a^2 - x^2} = \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} x \sqrt{a^2 - x^2}.$$

$$\boxed{\begin{array}{l} \sin\theta = \frac{x}{a} \\ x = a \sin\theta \\ \theta = \sin^{-1}\left(\frac{x}{a}\right) \end{array}}$$