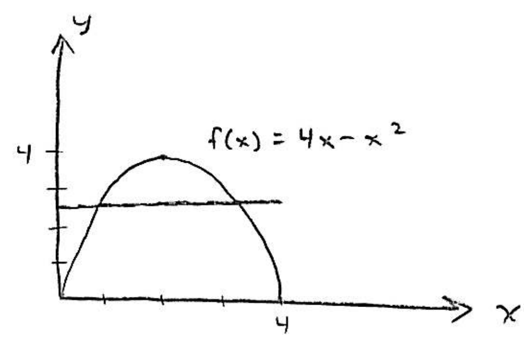
MATH 2 SOLUTIONS TO PROBLEM SET #13

SECTION 6.5 : AVERAGE VALUE OF A FUNCTION

(1.) $f(x) = 4x - x^2$ ON $[0, 4]$.

$$f_{ave} = \frac{1}{4}\int_0^4 4x - x^2 \, dx = \frac{1}{4}\left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{4}\left[\left(32 - \frac{64}{3}\right) - 0\right] = 8 - \frac{16}{3} = \boxed{\frac{8}{3}}.$$



(3.) $g(x) = \sqrt[3]{x}$ ON $[1, 8]$.

$$g_{ave} = \frac{1}{7}\int_1^8 \sqrt[3]{x} \, dx = \frac{1}{7}\int_1^8 x^{\frac{1}{3}} \, dx = \frac{1}{7}\left[\frac{3}{4}x^{\frac{4}{3}}\right]_1^8$$

$$= \frac{3}{28}(16 - 1) = \boxed{\frac{45}{28}}.$$

$\boxed{(4.)}$ $g(x) = x^2\sqrt{1 + x^3}$, $[0, 2]$

$$g_{ave} = \frac{1}{2}\int_0^2 x^2\sqrt{1 + x^3} \, dx = \frac{1}{2}\int_1^9 \frac{1}{3}\sqrt{u} \, du$$

$$= \frac{1}{6}\int_1^9 u^{\frac{1}{2}} \, du = \frac{1}{6}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^9$$

$$= \frac{1}{9}(27 - 1) = \boxed{\frac{26}{9}}.$$

$$\boxed{\begin{aligned} \text{LET } u &= 1 + x^3 \\ du &= 3x^2 \, dx \\ \tfrac{1}{3} du &= x^2 \, dx \end{aligned}}$$

(9.) $f(x) = (x-3)^2$ ON $[2, 5]$

(a.) $f_{ave} = \frac{1}{3}\int_2^5 (x-3)^2\,dx = \frac{1}{3}\int_2^5 x^2 - 6x + 9\,dx$

$= \frac{1}{3}\left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_2^5 = \frac{1}{3}\left[\left(\frac{125}{3} - 75 + 45\right) - \left(\frac{8}{3} - 12 + 18\right)\right]$

$= \frac{1}{3}\left[\frac{117}{3} - 30 - 6\right] = \frac{1}{3}(39 - 36) = \boxed{1}.$

OR, $f_{ave} = \frac{1}{3}\int_2^5 (x-3)^2\,dx \underset{\substack{\uparrow \\ u = x-3}}{=} \frac{1}{3}\int_{-1}^2 x^2\,dx$

$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^2 = \frac{1}{3}\left(\frac{8}{3} - \frac{-1}{3}\right) = \frac{1}{3}\left(\frac{9}{3}\right) = \boxed{1}.$

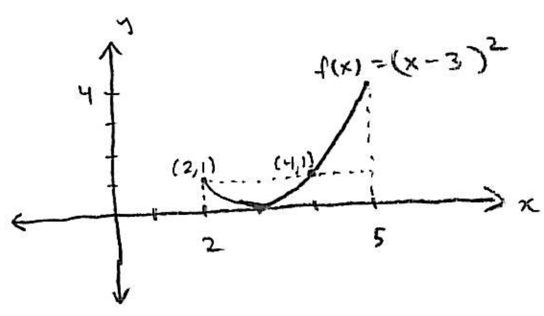
(b.) $f(c) = f_{ave}$

$(c-3)^2 = 1$

$c - 3 = \pm 1$

$c = 3 \pm 1 = \boxed{2 \text{ OR } 4}.$ (BOTH LIE IN $[2, 5]$.)

(c.)



$\boxed{(10.)}$ $f(x) = \sqrt{x}$ ON $[0, 4]$.

(a.) $f_{ave} = \frac{1}{4}\int_0^4 \sqrt{x}\, dx = \frac{1}{4}\int_0^4 x^{\frac{1}{2}}\, dx$

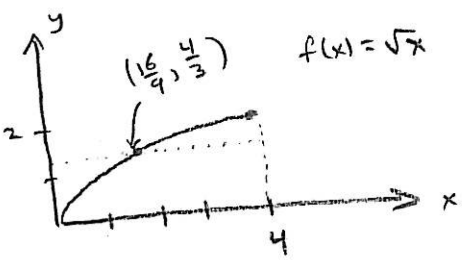
$= \frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{1}{6}(8-0) = \boxed{\frac{4}{3}}$.

(b.) $f(c) = f_{ave}$

$\sqrt{c} = \frac{4}{3}$

$c = \left(\frac{4}{3}\right)^2 = \boxed{\frac{16}{9}} = 1.777\ldots$

(c.)



(23.) MEAN VALUE THEOREM FOR INTEGRALS:

IF f IS CONTINUOUS ON $[a, b]$, THEN THERE EXISTS A NUMBER c IN $[a, b]$ SUCH THAT $f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx$.

PROOF: BY THE FUNDAMENTAL THEOREM PART I, $F(x) = \int_a^x f(t)\, dt$ IS CONTINUOUS ON $[a, b]$, DIFFERENTIABLE ON (a, b), AND SATISFIES $F'(x) = f(x)$. BY THE MEAN VALUE THEOREM FOR DERIVATIVES, IT FOLLOWS THAT THERE EXISTS c IN $[a, b]$ SUCH THAT $F'(c) = \frac{F(b) - F(a)}{b-a}$, OR EQUIVALENTLY, $f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$. ∎