SOLUTIONS TO PRACTICE EXAM II

(1.) $y^3 = y^5 \Rightarrow y^5 - y^3 = 0 \Rightarrow y^3(y^2-1) = 0 \Rightarrow y^3(y+1)(y-1) = 0$

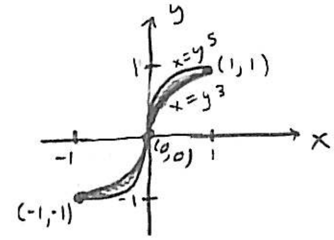
$\Rightarrow y = -1, 0,$ OR 1 SO POINTS OF INTERSECTION ARE

$(-1,-1), (0,0), (1,1)$. $y^5 > y^3$ FOR y BETWEEN $-1, 0$,

AND $y^5 < y^3$ FOR y BETWEEN $0, 1$, SO

$$A = \int_{-1}^{1} |y^5 - y^3|\, dy = \int_{-1}^{0} (y^5 - y^3)\, dy + \int_{0}^{1} (y^3 - y^5)\, dy$$

$$= \left[\frac{y^6}{6} - \frac{y^4}{4}\right]_{-1}^{0} + \left[\frac{y^4}{4} - \frac{y^6}{6}\right]_{0}^{1} = \frac{1}{12} + \frac{1}{12} = \boxed{\frac{1}{6}}.$$



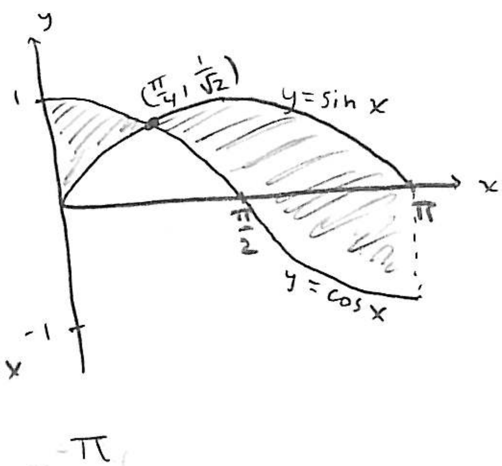
(2.) $\cos x = \sin x \Rightarrow 2\sin^2 x = 1$ (BY THE IDENTITY $\sin^2 x + \cos^2 x = 1$)

$\Rightarrow \sin^2 x = \frac{1}{2} \Rightarrow \sin x = \pm\frac{1}{\sqrt{2}}$. FOR x BETWEEN 0 AND π,

THIS OCCURS AT $x = \frac{\pi}{4}$: $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

AT $x = \frac{3\pi}{4}$: $\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$ BUT $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

SO THERE'S ONE POINT OF INTERSECTION, AT $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.



$\cos x > \sin x$ FOR x BETWEEN $0, \frac{\pi}{4}$

$\cos x < \sin x$ FOR x BETWEEN $\frac{\pi}{4}, \pi$,

SO $A = \int_0^{\pi} |\sin x - \cos x|\, dx$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi}$$

$$= (\sqrt{2} - 1) + \left(1 - (-\sqrt{2})\right) = \boxed{2\sqrt{2}}.$$

(3.) $y^2 - 4y + 3 = 3 \Rightarrow y^2 - 4y = 0 \Rightarrow y(y-4) = 0 \Rightarrow y = 0, 4$

SO PTS OF INTERSECTION ARE AT $(3,0)$, $(3,4)$.

($x = 3$ IS A VERTICAL LINE, $x = y^2 - 4y + 3$ IS A PARABOLA WHICH OPENS RIGHT, WITH VERTEX AT $(-1, 2)$.)

$y^2 - 4y + 3 < 3$ FOR y BETWEEN $0, 4$, SO

$$A = \int_0^4 |3 - (y^2 - 4y + 3)|\, dy = \int_0^4 4y - y^2 \, dy$$

$$= \left[2y^2 - \frac{y^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \boxed{\frac{32}{3}}.$$

(4.) $$g_{ave} = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\frac{\pi}{2}}} x \cos(x^2)\, dx$$

SUBSTITUTION:

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$x = 0 \Rightarrow u = 0$

$x = \sqrt{\frac{\pi}{2}} \Rightarrow u = \frac{\pi}{2}$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\frac{\pi}{2}} \frac{1}{2} \cos(u)\, du$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \cos(u)\, du$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\sin(u)\right]_0^{\pi/2} = \frac{1}{2}\sqrt{\frac{2}{\pi}}(1-0)$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} = \boxed{\frac{1}{\sqrt{2\pi}}}.$$

(5.) $$f_{ave} = \frac{2}{\pi} \int_{-\frac{\pi}{3}}^{\pi/6} \sec x \tan x \, dx = \frac{2}{\pi}\left[\sec x\right]_{-\pi/3}^{\pi/6}$$

$$= \frac{2}{\pi}\left[\frac{1}{\cos x}\right]_{-\pi/3}^{\pi/6} = \frac{2}{\pi}\left(\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} - \frac{1}{\frac{1}{2}}\right)$$

$$= \frac{2}{\pi}\left(\frac{2}{\sqrt{3}} - 2\right) = \boxed{\frac{4}{\pi}\left(\frac{1}{\sqrt{3}} - 1\right)}.$$

(6.) (a.) $h_{ave} = \frac{1}{2-1}\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx$

$= \left[-x^{-1}\right]_1^2 = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}.$

(b.) $h(c) = h_{ave} \Rightarrow \frac{1}{c^2} = \frac{1}{2} \Rightarrow c^2 = 2 \Rightarrow c = \pm\sqrt{2}$

$\Rightarrow \boxed{c = \sqrt{2}}$ (SINCE WE SEEK C BETWEEN 1 AND 2.)
(YES - MEAN VALUE THEOREM FOR INTEGRALS.)

(7.) (a.) $W = F \cdot d = (80\text{ lb})(30\text{ ft}) = \boxed{2{,}400\text{ ft-lb}}.$

(b.) $W = \int_0^{50} 50 - .6x\,dx = \left[50x - .3x^2\right]_0^{50}$

$= 2500 - .3(2500) = .7(2500) = \boxed{1750\text{ ft-lb}}$

(8.) $F(x) = kx$ (HOOKE'S LAW)

$F(.14) = k(.14) = 28$ so $k = \frac{28}{.14} = 200$

$F(x) = 200x$

$W = \int_{.1}^{.3} 200x\,dx = \left[100x^2\right]_{.1}^{.3} = 100\left[x^2\right]_{.1}^{.3}$

$= 100(.09 - .01) = 100(.08) = \boxed{8\text{ J}}.$

(9.) (a.) $F = ma = (30\text{ kg})(1.5\text{ m/s}^2) = \boxed{45\text{ N}}.$

(b.) $W = F \cdot d = (45\text{ N})(50\text{ m}) = \boxed{2{,}250\text{ J}}$

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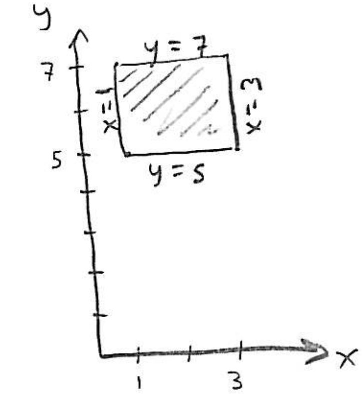
(10.) X-AXIS



(a.) WASHERS : $V = \pi \int_1^3 7^2 - 5^2 \, dx$

$= \pi \int_1^3 49 - 25 \, dx = \pi \int_1^3 24 \, dx$

$= \pi \left[24x \right]_1^3 = \boxed{48\pi}$

(b.) SHELLS : $V = 2\pi \int_5^7 y(3-1) \, dy$

$= 2\pi \int_5^7 2y \, dy = 2\pi \left[y^2 \right]_5^7 = 2\pi(24) = \boxed{48\pi}$.

(c.) $V = Ah$

$h = 2$

A = AREA OF BASE $= \pi \cdot 7^2 - \pi \cdot 5^2 = 24\pi$

SO $V = (24\pi)(2) = \boxed{48\pi}$.

(11.) Y-AXIS

(a.) WASHERS: $V = \pi \int_5^7 3^2 - 1^2 \, dy = \pi \int_5^7 8 \, dy = \pi \left[8y \right]_5^7 = \boxed{16\pi}$.

(b.) SHELLS: $V = 2\pi \int_1^3 x(7-5) \, dx = 2\pi \int_1^3 2x \, dx = 2\pi \left[x^2 \right]_1^3 = \boxed{16\pi}$.

(c.) $V = Ah$

$h = 2$

A = AREA OF BASE $= \pi \cdot 3^2 - \pi \cdot 1^2 = 8\pi$

$V = (8\pi)(2) = \boxed{16\pi}$.

(12.) THE REGION HUGS THE Y-AXIS MORE CLOSELY THAN THE X-AXIS, SO THE VOLUME OF THE SOLID OF REVOLUTION OBTAINED BY REVOLVING IT ABOUT THE Y-AXIS IS SMALLER (LESS VOLUME IS ENCLOSED).

(13.) SHELLS: $V = 2\pi \int_0^1 x e^{-x^2}\, dx$

SUBSTITUTION:

$u = -x^2$

$\frac{du}{dx} = -2x$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

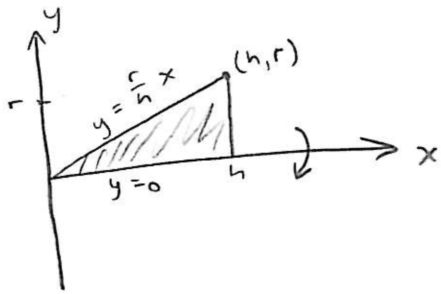
$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = -1$

$= 2\pi \int_0^{-1} -\frac{1}{2} e^u\, du$

$= \pi \int_{-1}^{0} e^u\, du = \pi \left[e^u \right]_{-1}^{0} = \boxed{\pi\left(1 - \frac{1}{e}\right)}.$

(14.)



$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx$

$= \pi \int_0^h \frac{r^2}{h^2} x^2\, dx$

$= \frac{\pi r^2}{h^2} \int_0^h x^2\, dx$

$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \boxed{\frac{1}{3}\pi r^2 h}.$

(15.) DISKS: $V = \pi \int_3^5 \left(\frac{1}{\sqrt{y}}\right)^2 dy = \pi \int_3^5 \frac{1}{y}\, dy$

$= \pi \left[\ln|y| \right]_3^5 = \pi(\ln 5 - \ln 3) = \boxed{\pi \ln\left(\frac{5}{3}\right)}.$

(16.) SHELLS: $V = 2\pi \int_0^{\pi^{1/3}} y\left(y e^{y^3} - y\cos(y^3)\right) dy$

$= 2\pi \int_0^{\pi^{1/3}} y^2 e^{y^3} - y^2 \cos(y^3)\, dy = 2\pi \int_0^{\pi} \frac{1}{3}\left(e^u - \cos(u)\right) du$

$= \frac{2\pi}{3} \int_0^{\pi} e^u - \cos(u)\, du$

$= \frac{2\pi}{3}\left[e^u - \sin u\right]_0^{\pi}$

$= \boxed{\frac{2\pi}{3}\left(e^{\pi} - 1\right)}.$

SUBSTITUTION:

$u = y^3$

$\frac{du}{dy} = 3y^2$

$du = 3y^2\, dy$

$\frac{1}{3}\, du = y^2\, dy$

$y = 0 \Rightarrow u = 0,\ y = \pi^{1/3} \Rightarrow u = \pi$

PRACTICE BONUS:

(a.) 13 – SHELLS. OTHERWISE, WE'D HAVE TO SOLVE $y = xe^{-x^2}$ FOR x IN TERMS OF y, JUST FOR STARTERS.

15 – DISKS. OTHERWISE, WE'D HAVE TO BREAK UP THE REGION INTO TWO PARTS.

16 – SHELLS. OTHERWISE WE'D HAVE TO SOLVE $x = ye^{y^3}$, $x = y\cos(y^3)$ FOR y IN TERMS OF x, JUST FOR STARTERS. WITH SHELLS, THE EXTRA FACTOR OF y MAKES FOR A NICE SUBSTITUTION.

(b.) $W = \int_0^{\pi} 100(x^2 + \sin x)\,dx$

$= 100\int_0^{\pi}(x^2 + \sin x)\,dx = 100\left[\frac{x^3}{3} - \cos x\right]_0^{\pi}$

$= 100\left[\left(\frac{\pi^3}{3} - (-1)\right) + (1)\right] = \boxed{100\left(\frac{\pi^3}{3} + 2\right)}.$