

Improper Integrals

The First Fundamental Theorem of Calculus, as we've discussed in class, goes as follows:

If f is continuous on the interval $[a, b]$ and F is a function for which $F'(t) = f(t)$, then

$$\int_a^b f(t)dt = F(b) - F(a).$$

An integral is improper when it doesn't meet the conditions given above. Specifically, one of two things could happen:

- The interval isn't of the form $[a, b]$ because one or the other (or both) of the endpoints of the interval is infinite.
- f is not continuous on the interval $[a, b]$ because there's a point on $[a, b]$ where f is not defined.

We'll refer to these as Type A and Type B improper integrals, respectively. There is one more case, where both of the problems above happen for the same integral; we call this a Type C improper integral. (We'll mostly be concerned with Type A, so don't worry too much about Types B and C.)

Solving a Type A Improper Integral

We can get around the problem of having $\pm\infty$ as an upper or lower bound by first replacing it with a variable. (Usually, this new variable is a , b , or c , but you can use just about whatever letter you like.) To see this, let's look at the integral $\int_0^\infty e^{-3x} dx$. Our first step is to replace the ∞ (here, I'll use b) to get a new integral:

$$\int_0^b e^{-3x} dx.$$

Strictly speaking, this isn't exactly the integral that we started with. But it *does* satisfy the criteria from the First Fundamental Theorem, so we can evaluate it in the usual way:

$$\begin{aligned} \int_0^b e^{-3x} dx &= -\frac{1}{3}e^{-3x} \Big|_0^b \\ &= -\frac{1}{3}e^{-3b} - \left(-\frac{1}{3}e^0\right) = -\frac{1}{3}e^{-3b} + \frac{1}{3}. \end{aligned}$$

At this point, we've found the area under the curve e^{-3x} on the interval $[0, b]$. We can evaluate the original integral by letting $b \rightarrow \infty$:

$$\int_0^\infty e^{-3x} dx = \lim_{b \rightarrow \infty} \left(\int_0^b e^{-3x} dx \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{3}e^{-3b} + \frac{1}{3} \right) = \lim_{b \rightarrow \infty} \left(-\frac{1}{3}e^{-3b} \right) + \frac{1}{3}.$$

We now calculate the limit by noticing that $e^{-3b} = \frac{1}{e^{3b}}$; as $b \rightarrow \infty$, the denominator goes to ∞ . This means that the fraction $\frac{1}{e^{3b}}$ goes to zero. This gives us the value of our integral:

$$\int_0^{\infty} e^{-3x} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{3}e^{-3b} \right) + \frac{1}{3} = 0 + \frac{1}{3} = \frac{1}{3}.$$

So the area under the curve e^{-3x} from 0 to ∞ is $\frac{1}{3}$. In this case, we say that the integral *converges* to $\frac{1}{3}$.

Of course, there's nothing to say that we'll always get a limit that isn't infinite. For example, let's try to evaluate $\int_{-\infty}^1 x^3 dx$. First of all, we have an infinite lower bound instead of an infinite upper bound. We get our new integral by replacing the lower bound (I'll replace it with a):

$$\int_a^1 x^3 dx.$$

Again, we've now rigged things so that we have an integral that satisfies the conditions of the First Fundamental Theorem. Evaluating our new integral, we get

$$\begin{aligned} \int_a^1 x^3 dx &= \frac{1}{4}x^4 \Big|_a^1 \\ &= \frac{1}{4}(1)^4 - \frac{1}{4}(a)^4 = \frac{1}{4} - \frac{1}{4}a^4. \end{aligned}$$

Now we evaluate our original integral by taking the limit as $a \rightarrow -\infty$:

$$\begin{aligned} \int_{-\infty}^1 x^3 dx &= \lim_{a \rightarrow -\infty} \left(\int_a^1 x^3 dx \right) \\ &= \lim_{a \rightarrow -\infty} \left(\frac{1}{4} - \frac{1}{4}a^4 \right) = \frac{1}{4} - \lim_{a \rightarrow -\infty} \left(\frac{1}{4}a^4 \right) \end{aligned}$$

Notice that since a is negative and approaching $-\infty$, a^4 is positive and approaching $+\infty$; thus, our limit is ∞ . In this case, we didn't get a finite value for our limit. So we say that the integral *diverges*.

In both of the integrals we just evaluated, we had either ∞ or $-\infty$ as one of our bounds of integration. If the integral has *both* ∞ and $-\infty$ as bounds, we can still evaluate it. Specifically, suppose that we want to evaluate the integral $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$. Here, our first case is to break up our interval (and our integral) into two pieces. We can pick really any point we want, but it's advisable to pick something that's easy to compute. I'll try breaking things apart at $x = 0$. This means that we can take

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \int_{-\infty}^0 \frac{1}{x^2+1} dx + \int_0^{\infty} \frac{1}{x^2+1} dx$$

Notice that this “breaking the interval apart” rule works even if the interval is finite and the integral isn’t improper. We’re just using it here as a tool to get nicer computations. By using this, we have that our original integral is a sum of two other ones. The two new integrals are of forms we’ve already seen, so we can evaluate them by replacing the infinities in the bounds; let’s do this as shown below:

$$\int_a^0 \frac{1}{x^2 + 1} dx + \int_0^b \frac{1}{x^2 + 1} dx$$

Again, you can call the new bounds whatever you like, but it’s important here that you use different letters in the two integrals so that you don’t incorrectly cancel things later on. We evaluate these two integrals by recalling that $\arctan(x)$ is an antiderivative of $\frac{1}{x^2+1}$:

$$\begin{aligned} & \int_a^0 \frac{1}{x^2 + 1} dx + \int_0^b \frac{1}{x^2 + 1} dx \\ &= \arctan(x) \Big|_a^0 + \arctan(x) \Big|_0^b \\ &= \arctan(0) - \arctan(a) + \arctan(b) - \arctan(0). \end{aligned}$$

The first thing to notice is that $\arctan(0) = 0$. The second thing is to recall that $\lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$ and that $\lim_{x \rightarrow -\infty} \arctan(x) = -\frac{\pi}{2}$. (Don’t worry if you don’t remember this fact about limits of \arctan ; this sort of thing won’t come up on the final exam.) In any case, we now take limits to evaluate our original integral:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx &= \int_{-\infty}^0 \frac{1}{x^2 + 1} dx + \int_0^{\infty} \frac{1}{x^2 + 1} dx \\ &= \lim_{a \rightarrow -\infty} \left(\int_a^0 \frac{1}{x^2 + 1} dx \right) + \lim_{b \rightarrow \infty} \left(\int_0^b \frac{1}{x^2 + 1} dx \right) \\ &= \arctan(0) - \lim_{a \rightarrow -\infty} (\arctan(a)) + \lim_{b \rightarrow \infty} (\arctan(b)) - \arctan(0) \\ &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi. \end{aligned}$$

It turns out that this integral converges, to π . Now, it’s entirely possible that the integral won’t converge in any nice way. The way to decide whether or not an integral of the $-\infty$ to ∞ variety diverges is to see whether or not either of the pieces diverge once you’ve broken it apart. If at least one of the two new integrals diverges, then your original one diverges.

Solving a Type B Improper Integral

Integrals of Type B are those which have a finite interval, but aren’t continuous everywhere on this interval. Specifically, there is a point in the interval somewhere for which the function isn’t defined. This is what I’ve been calling the “problem point.” If the problem point is at one or the other of the endpoints, we may still evaluate the integral by first inserting a new bound.

As an example, let's suppose we want to evaluate $\int_0^1 \frac{1}{x^2} dx$. The problem point here is at $x = 0$, since $\frac{1}{x^2}$ is not defined at this point. We get around this by replacing 0 with a as a bound of integration:

$$\int_a^1 \frac{1}{x^2} dx.$$

Technically, a is a small positive number. Think of this as “starting late” when constructing our interval. The point, though, is that we now have an integral that meets the conditions of the First Fundamental Theorem, and we can evaluate our new integral in the usual way:

$$\int_a^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_a^1 = -1 + \frac{1}{a}.$$

We can now recover our original integral by taking a limit:

$$\lim_{a \rightarrow 0^+} \left(-1 + \frac{1}{a} \right) = -1 + \lim_{a \rightarrow 0^+} \left(\frac{1}{a} \right)$$

Notice that we chose a to be a positive number; so when we let a approach zero, we're only taking a right-hand limit. It's now easy to see that the limit above diverges to ∞ . Consequently, our original integral diverges.

Another situation that could arise with Type B integrals is if the problem point isn't at the endpoints of our interval, but is somewhere in between. To see this, we consider the example of $\int_{-1}^1 \frac{1}{x^{2/3}} dx$. We make use of the same “breaking apart” trick that we did for certain integrals of Type A:

$$\int_{-1}^1 \frac{1}{x^{2/3}} dx = \int_{-1}^0 \frac{1}{x^{2/3}} dx + \int_0^1 \frac{1}{x^{2/3}} dx.$$

However, the choice of “break apart” point matters here. We need to choose the problem point (in this integral it's $x = 0$) as the point at which we break apart the integral. Notice that we now have two integrals of Type B; what's more, we have already discussed how to evaluate these types. Carrying out the process, we replace the problem endpoint in each integral:

$$\int_{-1}^b \frac{1}{x^{2/3}} dx + \int_a^1 \frac{1}{x^{2/3}} dx.$$

We finally have two integrals that meet the conditions of the First Fundamental Theorem. Let's evaluate them:

$$\begin{aligned} & \int_{-1}^b \frac{1}{x^{2/3}} dx + \int_a^1 \frac{1}{x^{2/3}} dx \\ &= 3x^{1/3} \Big|_{-1}^b + 3x^{1/3} \Big|_a^1 \\ &= 3b^{1/3} - 3(-1)^{1/3} + 3(1)^{1/3} - 3a^{1/3} \\ &= 3b^{1/3} + 3 + 3 - 3a^{1/3} = 3b^{1/3} - 3a^{1/3} + 6. \end{aligned}$$

And lastly, we take limits as $b \rightarrow 0^-$ and as $a \rightarrow 0^+$:

$$\int_{-1}^1 \frac{1}{x^{2/3}} dx = \lim_{b \rightarrow 0^-} \left(\int_{-1}^b \frac{1}{x^{2/3}} dx \right) + \lim_{a \rightarrow 0^+} \left(\int_a^1 \frac{1}{x^{2/3}} dx \right).$$

$$= \lim_{b \rightarrow 0^-} (3b^{1/3}) - \lim_{a \rightarrow 0^+} (3a^{1/3}) + 6.$$

But both of these limits are zero, since they both involve positive powers of a and b . Consequently, 6 is all that's left. And thus,

$$\int_{-1}^1 \frac{1}{x^{2/3}} dx = 6.$$

Final Comments

Notice that we haven't discussed how to test the convergence of improper integrals other than to make an attempt at evaluating integrals. What ends up happening with all of the above examples is that either we (a) find a value, or (b) find that the integral diverges.