

Math 2 – Winter 2006

Project 3 – Fun With Integration!

(Integrals involving inverse trigonometric functions, integrals by u-substitution, by parts, or a combination)

Instructions: Working in groups of two or three, complete problems 1-9. In addition, choose 3 of the challenging problems to do (problems 10-15). Your solutions are due on Monday, February 27, at the start of class. While you are encouraged to complete this project in groups, each person is responsible for handing in their own solutions.

1. $\int \theta \cos \theta \, d\theta$

2. $\int t e^{-t} \, dt$

3. $\int (2\theta + 1) \sin(\theta^2 + \theta + 43) \, d\theta$

4. $\int \sqrt{\frac{4+x}{4-x}} \, dx$

5. $\int_0^{\pi/6} \sin^2 \theta \cos \theta \, d\theta$

6. $\int t^4 \ln t \, dt$

7. $\int_0^{\pi/4} z \cos(2z) \, dz$

8. $\int w 3^w \, dw$

9. $\int_1^2 \frac{\ln t}{t^3} \, dt$

* = A more challenging integral

10. * $\int x^5 \sin(x^3) dx$

11. * $\int_0^{\pi/3} \cos^3 \theta d\theta$

12. * $\int e^x \cos x dx$

13. * $\int \frac{e^{2x}}{1+e^{2x}} dx$

14. * $\int_1^4 \ln(\sqrt{x}) dx$

15. * $\int (\ln z)^2 dz$

Integration Techniques

When you come across an integral, it might not be clear where to begin. Sometimes, it's as simple as doing a reverse power rule; other times, you may need to play around with it to get things into a recognizable form. The integration techniques given below, while certainly not exhaustive, should help you get an intuition for computing antiderivatives.

1. Multiply by a conjugate

This technique is usually used to clear up a fraction where a messy expression (possibly involving a square root) is getting in the way of the computation. An example of this is the integral

$$\int \sqrt{\frac{1+t}{1-t}} dt.$$

To simplify the expression, we multiply the top and bottom of the fraction by the conjugate of $\sqrt{1-t}$, which is $\sqrt{1+t}$:

$$\int \sqrt{\frac{1+t}{1-t}} dt = \int \sqrt{\frac{(1+t)(1+t)}{(1-t)(1+t)}} dt = \int \sqrt{\frac{(1+t)^2}{1-t^2}} dt = \int \frac{1+t}{\sqrt{1-t^2}} dt.$$

We can now separate this into two integrals:

$$\int \frac{1}{\sqrt{1-t^2}} dt + \int \frac{t}{\sqrt{1-t^2}} dt.$$

The first integral can be seen to have $\arcsin(t)$ as an antiderivative. The second one, while it is more difficult, can be seen to have $-\sqrt{1-t^2}$ as an antiderivative. (For now, we'll take this as a given; you'll see techniques for evaluating integrals like $\int \frac{t}{\sqrt{1-t^2}} dt$ in the coming weeks.) In any case, our integral then evaluates as

$$\int \sqrt{\frac{1+t}{1-t}} dt = \arcsin(t) - \sqrt{1-t^2} + C.$$

2. Simplify the expression and distribute the integral over sums

This technique is usually used to clear up a fraction. An example of this is the integral

$$\int \frac{(3^t + 1)^2}{3^t} dt.$$

To simplify the expression, we multiply out the top of the fraction $(3^t + 1)^2$ to get $(3^t)^2 + 2(3^t) + 1 = 3^{2t} + 2(3^t) + 1$. Now we have:

$$\int \frac{3^{2t} + 2(3^t) + 1}{3^t} dt = \int \left(\frac{3^{2t}}{3^t} + \frac{2(3^t)}{3^t} + \frac{1}{3^t} \right) dt.$$

Now, distributing the integral sign over sums and simplifying we have:

$$\int \left(\frac{3^{2t}}{3^t} \right) dt + \int \left(\frac{2(3^t)}{3^t} \right) dt + \int \left(\frac{1}{3^t} \right) dt = \int 3^t dt + \int 2 dt + \int 3^{-t} dt.$$

Hence, our answer is:

$$\int \frac{(3^t + 1)^2}{3^t} dt = \frac{3^t}{\ln(3)} + 2t - \frac{3^{-t}}{\ln(3)} + C.$$

3. Simplify the expression using long division and break apart sums

This is another technique that can be used to clear up a fraction. For example, the integral

$$\int \frac{(t+1)^2}{t+2} dt$$

may be computed by first noticing that $(t+1)^2 = t^2 + 2t + 1$ and then using long division to see that $t^2 + 2t + 1 = t + \frac{1}{t+2}$ and integrating this expression. Now we have

$$\int \frac{(t+1)^2}{t+2} dt = \int \left(t + \frac{1}{t+2} \right) dt.$$

Breaking apart the sum gives us

$$\int \left(t + \frac{1}{t+2} \right) dt = \int t dt + \int \frac{1}{t+2} dt.$$

Hence,

$$\int t dt + \int \frac{1}{t+2} dt = \frac{t^2}{2} + \ln(t+2) + C.$$