THE RECURSION THEOREM (DRAFT 1)

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Kleene's Recursion Theorem, though provable in only a few lines, is fundamental to computability theory and allows strong self-reference in proofs. It is a fixed-point theorem in the sense that it asserts for any total computable function f, there is a number n such that n and $f(n)$ code the same partial computable function (though we need not have $f(n) = n$). [brief outline of paper here]

The S-m-n Theorem and all versions of the Recursion Theorem are attributed to Kleene (see Soare [6]); the Relativized S-m-n Theorem is not attributed to anyone. [look up more here]

We begin with some background. [φ_e, equality for partial functions, anything else needed?]

be sure to comment explicitly on indices.

The basic theorem needed to prove the Recursion Theorem and its variants is the following, known as the S-m-n Theorem or the parametrization theorem.

Theorem 1 (S-m-n Theorem, Kleene). *Given m, n, there is a primitive recursive one-to-one function S^m_n such that for all e, all n-tuples $\bar{x}$, and all m-tuples $\bar{y}$,*

$$\varphi_{S^m_n(e,\bar{x})}(\bar{y}) = \varphi_e(\bar{x}, \bar{y}).$$

sketch the proof

Theorem 2 (Recursion or Fixed-Point Theorem, Kleene). *Suppose that f is a total computable function; then there is a number n such that $\varphi_n = \varphi_{f(n)}$. Moreover, n is computable from an index for f.*

introduce what you're doing here

If we could guarantee $f(\varphi_x(x))\downarrow$, then using the slightly circular choice of x as the index of $f \circ \varphi_x$ we would have $f(\varphi_x(x)) = (f \circ \varphi_x)(x) = \varphi_x(x)$, and so the functions indexed by $f(\varphi_x(x))$ and $\varphi_x(x)$ would be the same because those values would be equal. However, there is no guarantee of halting for $f(\varphi_x(x))$, and for a function such as $f(n) = n + 1$ we must have divergence. However, we may define a function on two inputs that mimics the desired function:

$$\varphi_e(x, y) = \begin{cases} \varphi_{f(\varphi_x(x))}(y) & \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Since I reverse-engineered this from the final version, it is a bit more polished than a first draft needs to be, but I hope it gets the idea across.

By the *S-m-n* Theorem 1, this function is equal to $\varphi_{s(x)}(y)$ for a total computable function s (technically the function produced by the *S-m-n* Theorem takes e as an input, but e is fixed and hence we ignore it). The key fact is that if $\varphi_x(x)\uparrow$, $s(x)$ will index a function that diverges everywhere, but will still be defined. what will? clarify.

|| * see below

Proof of Theorem 2. By the *S-m-n* Theorem there is a total computable function $s(x)$ such that for all x and y

$$\varphi_{f(\varphi_x(x))}(y) = \varphi_{s(x)}(y).$$

Let m be any index such that φ_m computes the function s; note that s and hence m are computable from an index for f. Rewriting the statement above yields

$$\varphi_{f(\varphi_x(x))}(y) = \varphi_{\varphi_m(x)}(y).$$

Then, putting $x = m$ and letting $n = \varphi_m(m)$ (which is defined because s is total), we have

$$\varphi_{f(n)}(y) = \varphi_{f(\varphi_m(m))}(y) = \varphi_{s(m)}(y) = \varphi_{\varphi_m(m)}(y) = \varphi_n(y)$$

as required. □

From the Recursion Theorem we obtain the immediate corollary that there are numbers n, m such that $\varphi_n = \varphi_{n+1}$ and $\varphi_m = \varphi_{2m}$, and we may continue in this manner for any total computable function. There are also the following corollaries.

(outline of) proof?

Corollary 3. *If f is a total computable function then there are arbitrarily large numbers n such that $\varphi_{f(n)} = \varphi_n$.*

Corollary 4. *If $f(x, y)$ is any computable function there is an index e such that $\varphi_e(y) = f(e, y)$.*

[examples of use of last corollary]

Many of the uses of the Recursion Theorem in computability-theoretic constructions can be summed up as building a Turing machine using the index of the finished machine. The construction will have early on a line something like "We construct a partial computable function ψ and assume by the Recursion Theorem that we have an index e for ψ." The construction, which is computable, is itself the function for which we seek a fixed point. When the construction is given the input e to be interpreted as the index of a partial computable function, it can use e to produce e', which is an index of the function ψ it is trying to build. The Recursion Theorem says the construction will have a fixed point, some i such that i and i' both index the same function, which must be careful! need to worry about uniformity

ψ. Furthermore this fixed point will be *computable* from an index for the construction itself.

Our first extension of the Recursion Theorem gives a fixed point of sorts for functions of two inputs.

Theorem 5 (Recursion Theorem with Parameters, Kleene)**.** *If $f(x,y)$ is a total computable function, then there is a total computable function $n(y)$ such that $\varphi_{n(y)} = \varphi_{f(n(y),y)}$ for all y.*

expand this. you'll use * above again, so may want to make it a separate comment right after S-m-n.

Proof. By the S-m-n Theorem there is a total computable function $d(x,y)$ such that

$$\varphi_{d(x,y)}(z) = \begin{cases} \varphi_{\varphi_x(x,y)}(z) & \text{if } \varphi_x(x,y)\downarrow; \\ \uparrow & \text{otherwise.} \end{cases}$$

Since f and d are both partial computable, there is an index v such that $\varphi_v(x,y) = f(d(x,y),y)$. Then $n(y) = d(v,y)$ is a fixed point for f, since

$$\varphi_{n(y)} = \varphi_{d(v,y)} = \varphi_{\varphi_v(v,y)} = \varphi_{f(d(v,y),y)} = \varphi_{f(n(y),y)}.$$

□

[Soare has a comment about replacing total f with partial ψ, put a description in here?] sure. Also: if this extends the original theorem, it should imply it. comment on that.

The second generalization of the Recursion Theorem we will include is the Relativized Recursion Theorem, which also allows parameters. [Describe relativization here.] yes

Theorem 6 (Relativized S-m-n Theorem)**.** *For every $m, n \geq 1$ there exists a one-to-one computable function S^m_n of $m+1$ variables so that for all sets $A \subseteq \mathbb{N}$ and for all $e, y_1, \ldots, y_m \in \mathbb{N}$,*

$$\varphi^A_{S^m_n(e,y_1,\ldots,y_m)}(z_1,\ldots,z_n) = \varphi^A_e(y_1,\ldots,y_m,z_1,\ldots,z_n).$$

[Describe proof and talk about the computability of the smn function and fixed point]

good. highlight the distinction from the usual results of relativizing to A (A-comp.)

Theorem 7 (Relativized Recursion Theorem (with Parameters), Kleene)**.** *Let $A \subseteq \mathbb{N}$. If $f(x,y)$ is an A-computable function, then there is a* computable *function $n(y)$ such that $\varphi^A_{n(y)} = \varphi^A_{f(n(y),y)}$ for all y. Moreover, n does not depend on A; namely, if $f(x,y) = \varphi^A_e(x,y)$, $n(y)$ can be found uniformly in e.*

as above in 5

Proof. By the Relativized S-m-n Theorem there is a total computable function $d(x,y)$ such that

$$\varphi^A_{d(x,y)}(z) = \begin{cases} \varphi^A_{\varphi_x(x,y)}(z) & \text{if } \varphi_x(x,y)\downarrow; \\ \uparrow & \text{otherwise.} \end{cases}$$

Since f and d are both computable in A, there is an index v such that $\varphi_v^A(x,y) = f(d(x,y),y)$. Then $n(y) = d(v,y)$ is a fixed point for f, since

$$\varphi_{n(y)}^A = \varphi_{d(v,y)}^A = \varphi_{\varphi_v^A(v,y)}^A = \varphi_{f(d(v,y),y)}^A = \varphi_{f(n(y),y)}^A.$$

□

Our final result is an application of the Relativized Recursion Theorem to the structure of Turing degrees.

might refresh readers on what the degrees are

Definition 8. The *Turing jump* of a set A, denoted A', is the Halting Set relativized to A. That is, $A' = \{e : \varphi_e^A(e)\downarrow\}$.

If $A \leq_T B$, then $A' \leq_T B'$ (and hence the jump is a well-defined operation on degrees), but it may be that $A <_T B$ and $A' \equiv_T B'$. We recall that the degree of computable sets is denoted $\mathbf{0}$ and hence the degree of the Halting Set is $\mathbf{0}'$. All degrees below $\mathbf{0}'$ must have jumps between $\mathbf{0}'$ and $\mathbf{0}''$. Degrees on the upper and lower extremes are called *high* and *low*, respectively. The following definition generalizes the notions of lowness and highness.

Definition 9. For each $n > 0$, define a degree $\boldsymbol{a} \leq \mathbf{0}'$ to be low$_n$ (high$_n$) if $\mathbf{0}^{(n)} = \boldsymbol{a}^{(n)}$ ($\boldsymbol{a}^{(n)} = \mathbf{0}^{(n+1)}$). A set A is low$_n$ (high$_n$) exactly when deg(A) is. We use low$_n$ and high$_n$ also to denote the collection of all low$_n$ or high$_n$ degrees. For convenience, we set low$_0 = \{\mathbf{0}\}$ and high$_0 = \{\mathbf{0}'\}$.

explain this a bit?

Note that low$_n \subseteq$ low$_{n+1}$ and high$_n \subseteq$ high$_{n+1}$. We state without proof that this containment is proper; the result is a corollary of the Jump Theorem 12. All proofs omitted below may be found in Soare [6].

Proposition 10. *For all $n \in \mathbb{N}$,* low$_n \neq$ low$_{n+1}$ *and* high$_n \neq$ high$_{n+1}$.

In some sense the low degrees are "close to" computable, and the high degrees are "close to" complete. The hierarchy of low$_n$ and high$_n$ degrees gradually carves out more and more of the c.e. degrees as n increases: low$_1$ (or just low) degrees are near $\mathbf{0}$, low$_2$ degrees come a little further up, low$_3$ a little further up yet; meanwhile the high$_n$ degrees are creeping down from near $\mathbf{0}'$. They can't overlap, but do they meet in the middle? Another corollary of the Jump Theorem 12, which uses the Relativized Recursion Theorem, says no, there is a gap.

a little odd to say it like this since you aren't the originator of the question

[look up citations for the below]

Proposition 11 (Martin, Lachlan, Sacks). *There is an intermediate c.e. degree* $\boldsymbol{a}$. *That is,* $\mathbf{0}^{(n)} < \boldsymbol{a}^{(n)} < \mathbf{0}^{(n+1)}$ *for all* n.

The proof requires the Sacks Jump Theorem, stated below.

Theorem 12 (Sacks Jump Theorem [4]). *Suppose we are given sets S and C such that $\emptyset' \leq_T S$, S is c.e. in $\emptyset'$, and $\emptyset <_T C \leq_T \emptyset'$. Then there exists a noncomputable c.e. set A such that $A' \equiv_T S$ and $C \not\leq_T A$. Furthermore, an index of A can be found uniformly from indices for S and C.*

In other words, if S *could* be the jump of a c.e. set, then up to Turing equivalence it is. [more explanation here.] This theorem is proved by an infinite injury construction. We will actually need the relativized version of this theorem, where for some set Y, $\emptyset$ is replaced by Y; therefore c.e. becomes Y-c.e. and computable becomes Y-computable. Sacks noted that in fact we can include another set D in the theorem; as long as D is c.e. (or Y-c.e.), $D' \leq_T S$, and $C \not\leq_T D$, we can also ensure $D \leq_T A$ and keep the uniformity of the theorem.

probably more streamlined to alter the theorem itself to incorporate these extensions.

Finally, we give the proof of the existence of an intermediate degree, as given in Soare [6] [to be expanded on!]. very much so

• what are S, C, D

• why is a intermediate

Proof of 11 (*Sacks* [5]). The uniformity of the Jump Theorem 12, combined with Sacks' observation stated after the theorem, and both relativized to Y gives a computable function q such that for all $x \in \mathbb{N}$ and $Y \subseteq \mathbb{N}$,

$$Y <_T W^Y_{q(x)} <_T Y' \quad \text{and} \quad (W^Y_{q(x)})' \equiv_T (W^{Y'}_x) \oplus Y'.$$

remind us what $\oplus$ is

Now apply the Relativized Recursion Theorem 7 to obtain a fixed point n such that $W^Y_{q(n)} = W^Y_n$ for all $Y \subseteq \mathbb{N}$. Define $\boldsymbol{a} = \deg(W^{\emptyset}_n)$. $\square$

References

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[3] Martin, D.A. On a question of G.E. Sacks. *Journal of Symbolic Logic* **31**(1966): 66–69.

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