What you learned in Math 28

Rosa C. Orellana

Chapter 1 - Basic Counting Techniques

Sum Principle

- If we have a partition of a finite set S, then the size of S is the sum of the sizes of the blocks of the partition. In other words, if $S = A_1 \cup A_2 \cup \cdots A_n$ where $A_i \cap A_j = \emptyset$ (mutually disjoint), then

$$|S| = |A_1| + |A_2| + \cdots + |A_n|$$

- This principle gives rise to an important counting technique. If you cannot count all the elements in the set at once, partition it into blocks that you can count.

Product Principle

- If we have a partition of a finite set S into m blocks, each of size n, then the size of S is mn.

- **General Product Principle:** If we make a sequence of m choices for which
  - there are k_1 possible first choices, and
  - for each way of making the first $i-1$ choices, there are k_i ways to make the i-th choose, then

  there are $\prod_{i=1}^{m} k_i = k_1 k_2 \cdots k_m$ ways to make the sequence of choices.

Applications of basic counting principles

- The number of functions from an m-element set to an n-element is n^m.

- The number of k**-element permutations of an** n**-element set** is $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$. This is also the number of one-to-one functions from a k-element set to an n-element set.

- The number of **permutations of an** n**-element set** is $n!$. (This is the number of bijections from n-element set to an n-element set).

- There are 2^n binary sequences of length n.

The bijection principle

- Two sets have the same size if and only if there is a bijection between them.

Applications of he bijection principle

- We proved that there are 2^n subsets of an n-element set by showing that there is a bijection between binary sequences of length n and subsets of an n-element set.

- $\binom{n}{k}$ is defined as the number of k-subsets of an n-element set.

- Using bijections and sum principle we proved that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

- Using a bijection we also showed $\binom{n}{k} = \binom{n}{n-k}$.

- We also proved that there is a bijection from subsets of an n-element set and functions from n-element set to a 2-element set.

The quotient principle

- If we partition a set P of size p into q blocks, each of size r, then the number of blocks is $q = p/r$.

- In this type of problems we are interested in counting the number of blocks rather than the number of elements in P.

- The main example here was the seating arrangements in a round table with n seats. Here P had $n!$ and each block had n elements, so the number of blocks corresponded to distinct seating arrangements.

- We also proved that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Applications to Lattice Paths and Catalan Paths

- The number of lattice paths from $(0,0)$ to (m,n) that only use right and up steps is $\binom{m+n}{n}$.

- We also proved that the number of **Catalan paths** is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Applications to Binomial Theorem

- **Binomial Theorem:** $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
- For this reason $\binom{n}{k}$ is called a **binomial coefficient**.
- Using this theorem we can prove identities such as

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

- We also proved that the number of even size subsets is equals to the number of odd size subsets.

The Pigeonhole Principle

- If we partition a set with more than n elements into n parts, then at least one part has more than one element.

- **Generalized pigeonhole principle:** If we partition a set with more than kn elements into n blocks, then at least one block has at least $k + 1$ elements.

- We used the pigeonhole for Ramsey problems such as in a group of 6 people there are at lest 3 that know each other or at least 3 that do not know each other.

Chapter 2 - Induction Problems

The Principle of Mathematical Induction

- The principle of mathematical induction: In order to prove a statement about an integer n, if we can

  1. Prove the statement when $n = b$, for some fixed integer b, and
  2. Show that the truth of the statement for $n = k - 1$ implies that truth of the statement for $n = k$ whenever $k > b$,

  then we can conclude the statement is true for all integers $n \geq b$.

Applications of the Principle of Mathematical Induction

- Using PMI we proved the binomial theorem again.

- Using PMI we proved the formula for the binomial coefficient: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Applications of the Principle of Mathematical Induction to proving recurrences

- A **recurrence** is an equation that expresses the nth term of a sequence a_n in terms of values a_i for $i < n$.

- You learned terminology related to recurrences: linear recurrence, driving function, homogeneous recurrences, constant coefficient linear recurrences.

- Example 1: The tower of Hanoi Problem - Found the recurrence and proved a solution using induction.

- Example 2: Drawing Circles on the plane - Found the recurrence and proved a solution using induction.

Solving first order linear recurrences

- A first order linear recurrence looks like: $a_n = ba_{n-1} + d$ where b and d are constants.

- The solution of this recurrence in terms of initial value a_0 and b and d is

$$a_n = a_0 b^n + d\frac{1 - b^n}{1 - b}$$

assuming $b \neq 0$.

The Principle of Strong Mathematical Induction

- In order to prove a statement about an integer n if we can

    1. Prove our statement when $n = b$, and

    2. Prove that the statement we get with $n = b$, $n = b + 1, \ldots, n = k - 1$ imply the statement with $n = k$,

  then our statement is true for all integers $n \geq b$.

Applications to Graph Theory

- You proved by induction that the sum of the degrees of the vertices is twice the number of edges.
- You proved that the number of vertices of odd degree must be even in a graph.
- We learned that a tree is a connected graph without cycles.
- We proved by strong induction that the number of edges in a tree is one less than the number of vertices.
- We proved by strong induction that if a tree has more than one vertex, the minimum number of vertices of degree one is two.

Applications to Graph Theory

- The number of labeled trees with n vertices and is n^{n-2}. We proved this by establishing a bijection to Prüfer codes.

- Now we can answer questions like: What is the number of labelled trees with 3 vertices of degree 1? because we can translate the problem to counting sequences of length $n-2$ whose values range between 1 and n.

- A final application of induction involved a recurrence for counting the number of spanning trees in a graph. We showed that $\#G = \#(G-e) + \#(G/e)$.

- We also learned how to find the minimum cost spanning trees.

Chapter 3- Distribution Problems

Distributions of k distinct objects to n distinct recipients - Functions

- Without any conditions there are n^k ways to do this - the number of functions.
- If each recipient gets at most one $n^{\underline{k}}$ - k-element permutations (one-to-one functions)
- If each recipient gets at least one $n!S(k, n)$ - number of onto functions.
- If each gets exactly one $n!$ - number of bijections.

Distributions of k distinct objects to n distinct recipients - Order matters

- Without conditions, k-th rising factorial $n^{\overline{k}} = (n)(n+1)\cdots n+k-1)$ - ordered functions.
- Each gets at least one, $k!\binom{k-1}{n-1} = k^{\underline{n}}(k-1)^{\underline{k-n}}$ - ordered onto functions.
- We obtained these formulas by counting ways to place distinct books into shelves.

Distributions of k identical objects to n distinct recipients

- Without conditions, $\binom{n+k-1}{k}$ - multisets.
- Each gets at most one, $\binom{n}{k}$ - subsets.
- Each gets at least one, $\binom{k-1}{n-1}$- compositions.
- We obtained these solutions by thinking of identical books into shelves.

Distributions of k distinct objects to n identical recipients - Set partitions

- Without conditions, $B(k)$, the bell numbers - Total number of set partitions of k elements into any number of blocks.

- Recurrence for $B(k) = \sum_{j=0}^{k-1} \binom{k-1}{j} B(j)$.

- Each gets at least one, $S(k, n)$ the Stirling number of the second kind - number of set partitions of k elements into n blocks.

- Recurrence: $S(k, n) = S(k-1, n-1) + nS(k-1, n)$. We did not have time to find a closed formula, so we only have the recurrence to find values of $S(k, n)$.

Distributions of k distinct objects to n identical recipients - Order matters

- Each gets at least one, Lah numbers $L(k,n) = \frac{k!}{n!}\binom{k-1}{n-1}$- Broken permutations.

- Recurrence for Lah numbers:
  $L(k,n) = L(k-1,n-1) + (n+k-1)L(k-1,n)$.

Distributions of k identical objects to n identical recipients

- Without conditions, $P(k)$ which is the number of partitions of k.

- Each gets at least one, $P(k,n)$ which is the number of partitions of k into n parts.

- Recurrence for $P(k,n) = \sum_{i=1}^{n} P(k-n,i)$.

- We can identify partitions with Ferrers diagrams (Young diagrams).

- Two operations on partitions that allowed us to obtain identities: Conjugation and Complement.

Multinomial Theorem

- Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_n)^n = \sum \binom{n}{j_1, j_2, \ldots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

  where the sum is over all lists of integers $j_1, j_2, \ldots, j_n$ that sum to n.

- Recall that we can think of $\binom{k}{j_1, j_2, \ldots j_n}$ as the number of ways to label the elements of an k element list with n labels so that label i is used j_i times.

Stirling Numbers and Polynomials

- The Stirling numbers of the first kind occur as coefficients when we expand $x^{\underline{k}}$ into powers of x.

$$x^{\underline{k}} = \sum_{n=0}^{k} s(k,n)x^n$$

  We obtained a recurrence for these numbers:
  $s(k,n) = s(k-1,n-1) - (k-1)s(k-1,n)$.

- The Stirling numbers of the second kind occur as coefficients when we write x^n as a linear combinations of $x^{\underline{i}}$.

$$x^n = \sum_{j=0}^{k} S(k,j)x^{\underline{j}}$$

Lah Numbers and Polynomials

- The Lah numbers occur as coefficients when we write $x^{\overline{k}}$ as a linear combination of $x^{\underline{n}}$:

$$x^{\overline{k}} = \sum_{n=0}^{k} L(k,n) x^{\underline{n}}$$

- We also had

$$x^{\underline{k}} = \sum_{n=0}^{k} (-1)^{n-k} L(k,n) x^{\overline{n}}$$

Chapter 4- Generating Functions

Introduction to generating functions

- Given a sequence of $a_0, a_1, a_2, \ldots$, the power series

$$\sum_{i=0}^{\infty} a_i x^i$$

is called the generating function for the sequence.

- Our objective in using generating functions is to find a polynomial, rational function or a function that equals the power series and allows us to compute the coefficients in an efficient manner.

- We use this method to give another proof of a special case of the binomial theorem: $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$.

Generating functions for multisets

- We also use this method to find the generating function for multisets:

$$\frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$

Generating functions of partitions

- The generating function for the number of partitions, $P(n)$ is $\sum_{n=0}^{\infty} P(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}$
- The generating function for partitions with distinct parts: $\prod_{i=1}^{\infty} (1+q^i)$
- The generating function for partitions with odd parts: $\prod_{i=1}^{\infty} \frac{1}{1-q^{2i-1}}$
- The generating function for partitions that fit into an $m \times n$ rectangle: $\binom{m+n}{n}_q = \frac{[m+n]_q!}{[m]_q![n]_q!}$

Using generating functions to solve recurrences

- Given a recurrence we can use generating functions to find a solution via algebraic manipulations of the generating function.

- If the sequence a_n satisfies the recurrence $a_i = ba_{i-1} + d_i$ then if $b \neq 0$, the generating function for a_i is

$$\sum_{i=0}^{\infty} a_i x^i = \frac{a_0 - d_0 + \sum_{j=1}^{\infty} d_j x^j}{1 - bx}$$

and $a_n = b^n(a_0 + \sum_{i=1}^{n} b^{-i} d_i)$

Chapter 5 - Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion

Theorem A: Given a collection of n sets A_1, $A_2, \ldots A_n$. The size of their union is given by

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{S:S\subseteq[n],S\neq\emptyset} (-1)^{|S|-1}\left|\bigcap_{i\in S} A_i\right|$$

Theorem B: If we assume that the sets A_1, $A_2, \ldots A_n$ are contained in a set A. Then the size of the complement of the union in the set A is given by

$$\left|\overline{\bigcup_{i=1}^{n} A_i}\right| = \sum_{S:S\subseteq[n]} (-1)^{|S|}\left|\bigcap_{i\in S} A_i\right|$$

Counting Derangements

A **derangement**, f, is a bijection from $[n]$ to $[n]$ so that $f(i) \neq i$ for all $i \in [n]$.

Theorem : The probability that a random bijection is a derangement is

$$\sum_{k=0}^{n} (-1)^k \frac{1}{k!},$$

which tends to $1/e \sim 0.3678\ldots$ as $n \to \infty$.

The number of onto functions

Theorem : The number of onto functions (surjections) from $[k]$ to $[n]$ is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$$

Corollary: The Stirling number of the second kind is given by

$$S(k,n) = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$$