Some Notes on Abstraction and Rigor

Definitions

Definitions in mathematics are somewhat different from definitions in English. In natural language, the definition of a word is determined by the usage and may evolve. For example, "broadcasting" was originally just a way of sowing seed. Someone used it by analogy to mean spreading messages widely, and then it was adopted for radio and TV. For speakers of present-day English I doubt the original planting meaning is ever the first to come to mind.

In contrast, in mathematics we begin with the definition and assign a term to it as a shorthand. That term then denotes exactly the objects which fulfill the terms of the definition. To say something is "by definition impossible" has a rigorous meaning in mathematics: if it contradicts one of the properties of the definition, it cannot hold of an object to which we apply the term.

Mathematical definitions do not have the fluidity of natural language definitions. Sometimes mathematical terms are used to mean more than one thing, but that is a re-use of the term and not an evolution of the definition. Furthermore, mathematicians dislike that because it leads to ambiguity (exactly what is being meant by this term in this context?), which defeats the purpose of mathematical terms in the first place: to serve as shorthand for specific lists of properties.

Proofs

Let me try to set out some guidelines for writing proofs, in advance of the actual writing. Our proofs will bear little to no resemblance to the two-column proofs many of you met in high school geometry.

A proof is an object of convincing. It should be an explicit, specific, logically sound argument that walks step by step from the hypotheses to the conclusions. That is, avoid vagueness and leaps of deduction, and strip out irrelevant statements. Make your proof self-contained except for explicit reference to definitions or previous results (i.e., don't assume your reader is familiar with the theorems so you may use them without comment; instead say "by Theorem 2.5, ...").

While symbols often streamline mathematical writing, our proofs will be very verbal. I never want to see a proof which is just strings of symbols with only a few words. However, it can be clumsy and expand proofs out of readability to avoid symbols altogether. It is also important for specificity to assign symbolic names to (arbitrary) numbers and other objects to which you will want to refer. Striking the symbol/word balance will be one of our goals.

Your audience is a person who is familiar with the underlying definitions used in the statement being proved, but not the statement itself. For instance, it could be yourself after you learned the definitions, but before you had begun work on the proof. You do not have to put every tiny painful step in the write-up, but be careful about what you assume of the reader's ability to fill in gaps.

We will discuss with examples how one may insert small statements (I call it "foreshadowing" or "telegraphing") to make the proof much easier to follow. In particular, when working by contradiction or induction, it is important to let the reader know at the beginning.

Miscellaneous notes on writing for proofs:

There is a place for words like *would*, *could*, *should*, *might*, and *ought* in proofs, but they should be kept to a minimum. Most of the time the appropriate words are *has*, *will*, *does*, and *is*. This is especially important in proofs by contradiction. Since in such a proof you are assuming something which is not true, it may feel more natural to use the subjunctive, but that comes across as tentative. You assume some hypothesis; given that hypothesis other statements *are* or *are not* true. Be bold and let the whole contraption go up in flames when it runs into the statement it contradicts.

If a sentence seems strained, try turning it around. That is, try putting descriptive phrases before the noun they modify instead of after (or vice-versa), and putting clauses of the sentence into a different order. Rearranging several consecutive sentences may also be helpful. Do not fear to edit: the goal is a readable proof that doesn't require too much back-and-forth to understand.

And finally, one small thing: Unless otherwise stated, in the proof of a theorem the phrase "the theorem" refers to the theorem being proved. No need to use explanatory phrases (likewise for propositions, lemmas, corollaries, etc.).

Cautionary notes:

* If you have a definition before you of a particular concept and are asked to prove something about the concept, you must stick to the definition.

* Be wary of mentally adding words like *only*, *for all*, *for every*, or *for some* which are not actually there. Likewise, don't insert uniqueness where it's not asked for; if you are not asked to prove something is unique it may be because it's not.

* If you are asked to prove something holds of all objects of some type, you cannot pick a specific example and show the property holds of *that* object – it is not a proof that it works for *all*. Instead give a symbolic name to an arbitrary example and prove the property holds using only facts that are true of all objects of the given type.

* The theorems I give you to prove will not have redundant hypotheses. That is, all the hypotheses must be true for us to assert that the conclusion is always true. Therefore you must use all of the hypotheses in the proof.

* And finally, though this is indeed not English class, I will be checking your English. Small misspellings and the like will not be grounds for proof rejection, but sentence fragments and tortured grammar will be – make sure what your pronouns refer to is clear and that your verbs do not have confused objects.