Chapter 2

Combinatorial Applications of Induction

2.1 Some Examples of Mathematical Induction

In Chapter 1 (Problem 20), we used the principle of mathematical induction to prove that a set of size n has 2^n subsets. If you were unable to do that problem and you haven't yet read Appendix B, you should do so now.

2.1.1 Mathematical induction

The **principle of mathematical induction** states that

> In order to prove a statement about an integer n, if we can
>
> 1. Prove the statement when $n = b$, for some fixed integer b
> 2. Show that the truth of the statement for $n = k - 1$ implies the truth of the statement for $n = k$ whenever $k > b$,
>
> then we can conclude the statement is true for all integers $n \geq b$.

As an example, let us return to Problem 20. The statement we wish to prove is the statement that "A set of size n has 2^n subsets."

> Our statement is true when $n = 0$, because a set of size 0 is the empty set and the empty set has $1 = 2^0$ subsets. (This step of our proof is called a *base step.*)
>
> Now suppose that $k > 0$ and every set with $k-1$ elements has 2^{k-1} subsets. Suppose $S = \{a_1, a_2, \ldots a_k\}$ is a set with k elements. We partition the subsets of S into two blocks. Block B_1 consists of the subsets that do not contain a_n and block B_2 consists of the subsets that do contain a_n. Each set in B_1 is a subset of $\{a_1, a_2, \ldots a_{k-1}\}$, and each subset of $\{a_1, a_2, \ldots a_{k-1}\}$ is in B_1. Thus B_1 is the set of all subsets of $\{a_1, a_2, \ldots a_{k-1}\}$. Therefore by our assumption in the first sentence of this paragraph, the size of B_1 is 2^{k-1}. Consider the function from B_2 to B_1 which takes a subset of S including a_k and removes a_k from it. This function is defined on B_2, because every set in B_2 contains a_k. This function is onto, because if T is a set in B_1, then $T \cup \{a_k\}$ is a set in B_2 which the function sends to T. This function is one-to-one because if V and W are two different sets in B_2, then removing a_k from them gives two different sets in B_1. Thus we have a bijection between B_1 and B_2, so B_1 and B_2 have the same size. Therefore by the sum principle the size of $B_1 \cup B_2$ is $2^{k-1} + 2^{k-1} = 2^k$. Therefore S has 2^k subsets. This shows that if a set of size $k-1$ has 2^{k-1} subsets, then a set of size k has 2^k subsets. Therefore by the principle of mathematical induction, a set of size n has 2^n subsets for every nonnegative integer n.

The first sentence of the last paragraph is called the *inductive hypothesis.* In an inductive proof we always make an inductive hypothesis as part of proving that the truth of our statement when $n = k - 1$ implies the truth of our statement when $n = k$. The last paragraph itself is called the *inductive step* of our proof. In an inductive step we derive the statement for $n = k$ from the statement for $n = k - 1$, thus proving that the truth of our statement when $n = k - 1$ implies the truth of our statement when $n = k$. The last sentence in the last paragraph is called the *inductive conclusion.* All inductive proofs should have a base step, an inductive hypothesis, an inductive step, and an inductive conclusion.

There are a couple details worth noticing. First, in this problem, our base step was the case $n = 0$, or in other words, we had $b = 0$. However, in other proofs, b could be any integer, positive, negative, or 0. Second, our proof that the truth of our statement for $n = k - 1$ implies the truth of our statement for $n = k$ required that k be at least 1, so that there would be an element a_k we could take away in order to describe our bijection. However, condition (2) of the principle of mathematical induction only requires that we be able to prove the implication for $k > 0$, so we were allowed to assume $k > 0$.

2.1.2 Binomial coefficients and the Binomial Theorem

63. When we studied the Pascal Equation and subsets in Chapter 1, it may have appeared that there is no connection between the Pascal relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ and the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Of course you probably realize you can prove the Pascal relation by substituting the values the formula gives you into the right-hand side of the equation and simplifying to give you the left hand side. In fact, from the Pascal Relation and the facts that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, you can actually prove the formula for $\binom{n}{k}$ by induction. Do so.

64. Use the fact that $(x+y)^n = (x+y)(x+y)^{n-1}$ to give an inductive proof of the binomial theorem.

2.1.3 Inductive definition

You may have seen $n!$ described by the two equations $0! = 1$ and $n! = n(n-1)!$ for $n > 0$. By the principle of mathematical induction we know that this pair of equations defines $n!$ for all nonnegative numbers n. For this reason we call such a definition an **inductive definition**. An inductive definition is sometimes called a *recursive definition*. Often we can get very easy proofs of useful facts by using inductive definitions.

65. An inductive definition of a^n for nonnegative n is given by $a^0 = 1$ and $a^n = aa^{n-1}$.

    (a) Use this definition to prove the rule of exponents $a^{m+n} = a^m a^n$ for nonnegative m and n.

    (b) Use this definition to prove the rule of exponents $a^{mn} = (a^m)^n$.

66. Give an inductive definition of the summation notation $\sum_{i=1}^n a_i$. Use it to prove the distributive law
$$b\sum_{i=1}^n a_i = \sum_{i=1}^n ba_i.$$

2.1.4 Proving the general product principle (Optional)

We stated the sum principle as

> If we have a partition of a set S, then the size of S is the sum of the sizes of the blocks of the partition.

In fact, the simplest form of the sum principle says that the size of the sum of two disjoint (finite) sets is the sum of their sizes.

67. Prove the sum principle we stated for partitions of a set from the simplest form of the sum principle.

We stated the simplest form of the product principle as

> If we have a partition of a set S into m blocks, each of size n, then S has size mn.

In Problem 21 we gave a more general form of the product principle which can be stated as

> Let S be a set of functions f from $[n]$ to some set X. Suppose there are k_1 choices for $f(1)$. Suppose that for each choice of $f(1)$ there are k_2 choices for $f(2)$. In general, suppose that for each choice of $f(1)$, $f(2)$, $\dots f(i-1)$, there are k_i choices for $f(i)$. Then the number of functions in the set S is $k_1k_2\cdots k_n$.

68. If you weren't able to do so in Problem 21, prove the general form of the product principle from the simplest form of the product principle.

2.1.5 Double Induction and Ramsey Numbers

In Section 1.3.4 we gave two different descriptions of the Ramsey number $R(m,n)$. However if you look carefully, you will see that we never showed that Ramsey numbers actually exist; we merely described what they were and showed that $R(3,3)$ and $R(3,4)$ exist by computing them directly. As long as we can show that there is some number R such that when there are R people together, there are either m mutual acquaintances or n mutual strangers, this shows that the Ramsey Number $R(m,n)$ exists, because it is the smallest such R. Mathematical induction allows us to show that one such R is $\binom{m+n-2}{m-1}$. The question is, what should we induct on, m or n? In other words, do we use the fact that with $\binom{m+n-3}{m-2}$ people in a room there are at least $m-1$ mutual acquaintances or n mutual strangers, or do we use the fact that with at least $\binom{m+n-3}{n-2}$ people in a room there are at least m mutual acquaintances or at least $n-1$ mutual strangers? It turns out that we use both. Thus we want to be able to simultaneously induct on m and n. One way to do that is to use yet another variation on the principle of mathematical induction, the *Principle of Double Mathematical Induction.* This principle (which can be derived from one of our earlier ones) states that

> In order to prove a statement about integers m and n, if we can
>
> 1. Prove the statement when $m = a$ and $n = b$, for fixed integers a and b
> 2. Prove the statement when $m = a$ and $n > b$ and when $m > a$ and $n = b$ (for the same fixed integers a and b),
> 3. Show that the truth of the statement for $m = j$ and $n = k-1$ (with $j \geq a$ and $k > j$) and the truth of the statement for $m = j-1$ and $n = k$ (with $j > a$ and $k \geq b$) imply the truth of the statement for $m = j$ and $n = k$,
>
> then we can conclude the statement is true for all pairs of integers $m \geq a$ and $n \geq b$.

69. Prove that $R(m,n)$ exists by proving that if there are $\binom{m+n-2}{m-1}$ people in a room, then there are either at least m mutual acquaintances or at least n mutual strangers.

70. Prove that $R(m,n) \leq R(m-1,n) + R(m,n-1)$.

71. (a) What does the equation in Problem 70 tell us about $R(4,4)$?

    (b) Consider 17 people arranged in a circle such that each person is acquainted with the first, second, fourth, and eighth person to the right and the first, second, fourth, and eighth person to the left. can you find a set of four mutual acquaintances? Can you find a set of four mutual strangers?

(c) What is $R(4, 4)$?

72. (Optional) Can you prove the equation of Problem 70 by induction on $m + n$? If so, do so, and if not, explain where there is a problem in trying to do so.

73. (Optional) Prove the Principle of Double Mathematical Induction from the Principle of Mathematical Induction.

2.2 Recurrence Relations

We have seen in Problem 20 (or Problem 21 in the Appendix on Induction) that the number of subsets of an n-element set is twice the number of subsets of an $n-1$-element set.

74. Explain why it is that the number of bijections from an n-element set to an n-element set is equal to n times the number of bijections from an $(n-1)$-element subset to an $(n-1)$-element set. What does this have to do with Problem 26?

We can summarize these observations as follows. If s_n stands for the number of subsets of an n-element set, then

$$s_n = 2s_{n-1}, \tag{2.1}$$

and if b_n stands for the number of bijections from an n-element set to an n-element set, then

$$b_n = nb_{n-1}. \tag{2.2}$$

Equations 2.1 and 2.2 are examples of *recurrence equations* or *recurrence relations*. A **recurrence relation** or simply a **recurrence** is an equation that expresses the nth term of a sequence a_n in terms of values of a_i for $i < n$. Thus Equations 2.1 and 2.2 are examples of recurrences.

2.2.1 Examples of recurrence relations

Other examples of recurrences are

$$a_n = a_{n-1} + 7, \tag{2.3}$$

$$a_n = 3a_{n-1} + 2^n, \tag{2.4}$$

$$a_n = a_{n-1} + 3a_{n-2}, \text{ and} \tag{2.5}$$

$$a_n = a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1. \tag{2.6}$$

A **solution** to a recurrence relation is a sequence that satisfies the recurrence relation. Thus a solution to Recurrence 2.1 is $s_n = 2^n$. Note that $s_n = 17 \cdot 2^n$ and $s_n = -13 \cdot 2^n$ are also solutions to Recurrence 2.1. What this shows is that a recurrence can have infinitely many solutions. In a given problem, there is generally one solution that is of interest to us. For example, if we are interested in the number of subsets of a set, then the solution to Recurrence 2.1 that we care about is $s_n = 2^n$. Notice this is the only solution we have mentioned that satisfies $s_0 = 1$.

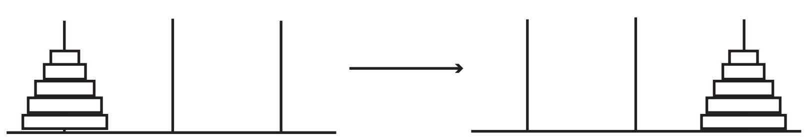
75. Show that there is only one solution to Recurrence 2.1 that satisfies $a_0 = 1$.

76. A first-order recurrence relation is one which expresses a_n in terms of a_{n-1} and other functions of n, but which does not include any of the terms a_i for $i < n - 1$ in the equation.

    (a) Which of the recurrences 2.1 through 2.6 are first order recurrences?

    (b) Show that there is one and only one sequence a_n that is defined for every nonnegative integer n, satisfies a first order recurrence, and satisfies $a_0 = a$ for some fixed constant a.

Figure 2.1: The Towers of Hanoi Puzzle



77. The "Towers of Hanoi" puzzle has three rods rising from a rectangular base with n rings of different sizes stacked in decreasing order of size on one rod. A legal move consists of moving a ring from one rod to another so that it does not land on top of a smaller ring. If m_n is the number of moves required to move all the rings from the initial rod to another rod that you choose, give a recurrence for m_n. (Hint: suppose you already knew the number of moves needed to solve the puzzle with $n-1$ rings.)

78. We draw n mutually intersecting circles in the plane so that each one crosses each other one exactly twice and no three have a boundary point in common. (As examples, think of Venn diagrams with two or three mutually intersecting sets.) Find a recurrence for the number r_n of regions into which the plane is divided by n circles. (One circle divides the plane into two regions, the inside and the outside.) Find the number of regions with n circles. For what values of n can you draw a Venn diagram showing all the possible intersections of n sets using circles to represent each of the sets?

2.2.2 Arithmetic Series

79. A child puts away two dollars from her allowance each week. If she starts with twenty dollars, give a recurrence for the amount a_n of money she has after n weeks and find out how much money she has at the end of n weeks.

80. A sequence that satisfies a recurrence of the form $a_n = a_{n-1} + c$ is called an *arithmetic progression*. Find a formula in terms of the initial value a_0 and the common difference c for the term a_n in an arithmetic progression and prove you are right.

81. A person who is earning \$50,000 per year gets a raise of \$3000 a year for n years in a row. Find a recurrence for the amount a_n of money the person earns over $n+1$ years. What is the total amount of money that the person earns over a period of $n+1$ years? (In $n+1$ years, there are n raises.)

82. An *arithmetic series* is a sequence s_n equal to the sum of the terms a_0 through a_n of of an arithmetic progression. Find a recurrence for the

sum s_n of an arithmetic progression with initial value a_0 and common difference c (using the language of Problem 80). Find a formula for general term s_n of an arithmetic series.

2.2.3 First order linear recurrences

Recurrences such as those in Equations 2.1 through 2.5 are called *linear recurrences*, as are the recurrences of Problems 77 and 78. A **linear recurrence** is one in which a_n is expressed as a sum of functions of n times values of (some of the terms) a_i for $i < n$ plus (perhaps) another function (called the *driving function*) of n. A linear equation is called *homogeneous* if the driving function is zero (or, in other words, there is no driving function). It is called a constant coefficient linear recurrence if the functions that are multiplied by the a_i terms are all constants (but the driving function need not be constant).

83. Classify the recurrences in Equations 2.1 through 2.5 and Problems 77 and 78 according to whether or not they are constant coefficient, and whether or not they are homogeneous.

84. As you can see from Problem 83 some interesting sequences satisfy first order linear recurrences, including many that have constant coefficients, have constant driving term, or are homogeneous. Find a formula for the general term a_n of a sequence that satisfies a constant coefficient first order linear recurrence $a_n = ba_{n-1} + d$ in terms of b, d, a_0 and n and prove you are correct. If your formula involves a summation, try to replace the summation by a more compact expression.

2.2.4 Geometric Series

A sequence that satisfies a recurrence of the form $a_n = ba_{n-1}$ is called a *geometric progression*. Thus the sequence satisfying Equation 2.1, the recurrence for the number of subsets of an n-element set, is an example of a geometric progression. From your solution to Problem 84, a geometric progression has the form $a_n = a_0 b^n$. In your solution to Problem 84 you may have had to deal with the sum of a geometric progression in just slightly different notation, namely $\sum_{i=0}^{n-1} db^i$. A sum of this form is called a **(finite) geometric series**.

85. Do this problem only if your final answer (so far) to Problem 84 contained the sum $\sum_{i=0}^{n-1} db^i$.

    (a) Expand $(1-x)(1+x)$? Expand $(1-x)(1+x+x^2)$. Expand $(1-x)(1+x+x^2+x^3)$.

    (b) What do you expect $(1-b)\sum_{i=0}^{n-1} db^i$ to be? What formula for $\sum_{i=0}^{n-1} db^i$ does this give you? Prove that you are correct.

In Problem 84 and perhaps 85 you proved an important theorem.

Theorem 2 *If $a_n = ba_{n-1} + d$, then $a_n = a_0b^n + d\frac{1-b^n}{1-b}$.*

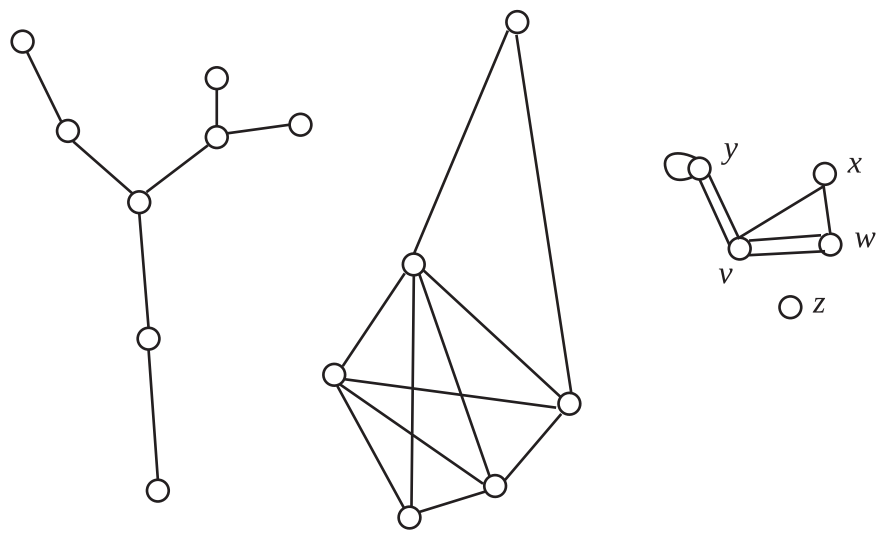
2.3 Trees

2.3.1 Undirected graphs

In Section 1.3.4 we introduced the idea of a directed graph. Graphs consist of vertices and edges. We describe vertices and edges in much the same way as we describe points and lines in geometry: we don't really say what vertices and edges are, but we say what they do. We just don't have a complicated axiom system the way we do in geometry. A **graph** consists of a set V called a vertex set and a set E called an edge set. Each member of V is called a *vertex* and each member of E is called an *edge*. Just as lines can connect points in geometry, edges can connect vertices in graph theory. We have one axiom like the axioms of geometry, namely, each edge connects two vertices. We draw pictures of graphs much like we draw pictures of geometric objects. In Figure 2.2 we show three pictures of graphs. Each circle in the figure represents a vertex; each line segment represents an edge. You will note that in the third graph we labelled the vertices; these labels are names we chose to give the vertices. We can choose names or not as we please. The third graph also shows that it is possible to have an edge that connects a vertex (like the one labelled y) to itself or it is possible to have two or more edges (like those between vertices v and y) between two vertices. The *degree* of a vertex is the number of times it appears as the endpoint of edges; thus the degree of y in the third graph in the figure is four.

86. The sum of the degrees of the vertices of a graph is related in a natural way to the number of edges. What is the relationship? Prove you are right. (Try to formulate your proof both with and without induction.)

Figure 2.2: Three different graphs



2.3.2 Walks and paths in graphs

A *walk* in a graph is an alternating sequence $v_0e_1v_1 \ldots e_iv_i$ of vertices and edges such that edge e_i connects vertices v_{i-1} and v_i. A graph is called connected if, for any pair of vertices, there is a walk starting at one and ending at the other.

87. Which of the graphs in Figure 2.2 is connected?

88. A *path* in a graph is a walk with no repeated vertices. Find the longest path you can in the third graph of Figure 2.2.

89. A *cycle* in a graph is a walk whose first and last vertex are equal with no other repeated vertices. Which graphs in Figure 2.2 have cycles? What is the largest number of edges in a cycle in the second graph in Figure 2.2? What is the smallest number of edges in a cycle in the third graph in Figure 2.2?

90. A connected graph with no cycles is called a **tree**. Which graphs, if any, in Figure 2.2 are trees?

2.3.3 Counting vertices, edges, and paths in trees

91. Draw some trees and on the basis of your examples, make a conjecture about the relationship between the number of vertices and edges in a tree. Prove your conjecture. (Hint: what happens if you choose an edge and delete it, but not its endpoints?)

92. What is the minimum number of vertices of degree one in a tree? What is it if the number of vertices is bigger than one? Prove that you are correct.

93. In a tree, given two vertices, how many paths can you find between them? Prove that you are correct.

94. How many trees are there on the vertex set $\{1, 2\}$? On the vertex set $\{1, 2, 3\}$? When we label the vertices of our tree, we consider the tree which has edges between vertices 1 and 2 and between vertices 2 and 3 different from the tree that has edges between vertices 1 and 3 and between 2 and 3. See Figure 2.3. How many (labelled) trees are there

Figure 2.3: These two trees are different

on four vertices? You don't have a lot of data to guess from, but try to guess a formula for the number of trees with vertex set $\{1, 2, \cdots, n\}$. (If you organize carefully, you can figure out how many labelled trees there are with vertex set $\{1, 2, 3, 4, 5\}$ to help you make your conjecture.) Given a tree with two or more vertices, labelled with positive integers, define a sequence of integers inductively as follows: If the tree has two vertices, the sequence consists of one entry, namely the vertex with the larger label. Otherwise, let a_1 be the lowest numbered vertex of degree 1 in the tree. Let b_1 be the unique vertex in the tree adjacent to a_1 and write down b_1. Then write down the sequence corresponding to the tree you get when you delete a_1 from the tree. (If you are unfamiliar with inductive (recursive) definition, you might want to write down some labelled trees on, say, ten vertices and construct the sequence b.) How long will the sequence be if it is applied to a tree with n vertices (labelled with 1 through n)? What can you say about the last member of the sequence of b_is? Can you tell from the sequence of b_is what a_1 is? Find a bijection between labelled trees and something you can "count" that will tell you how many labelled trees there are on n labelled vertices.

2.3.4 Spanning trees

Many of the applications of trees arise from trying to find an efficient way to connect all the vertices of a graph. For example, in a telephone network, at any given time we have a certain number of wires (or microwave channels, or cellular channels) available for use. These wires or channels go from a specific place to a specific place. Thus the wires or channels may be thought of as edges of a graph and the places the wires connect may be thought of as vertices of that graph. A tree whose edges are some of the edges of a graph G and whose vertices are all of the vertices of the graph G is called a **spanning tree** of G. A spanning tree for a telephone network will give us a way to route calls between any two vertices in the network.

95. Show that every connected graph has a spanning tree. Can you give two essentially different proofs (they needn't be completely different, but should be different in at least one significant aspect)?

2.3.5 Minimum cost spanning trees

Our motivation for talking about spanning trees was the idea of finding a minimum number of edges we need to connect all the edges of a communication network together. In many cases edges of a communication network come with costs associated with them. For example, one cell-phone operator charges another one when a customer of the first uses an antenna of the other. Suppose a company has offices in a number of cities and wants to put together a communication network connecting its various locations with high-speed computer communications, but to do so at minimum cost. Then it wants to take a graph whose vertices are the cities in which it has offices and whose edges represent possible communications lines between the cities. Of course there will not necessarily be lines between each pair of cities, and the company will not want to pay for a line connecting city i and city j if it can already connect them indirectly by using other lines it has chosen. Thus it will want to choose a spanning tree of minimum cost among all spanning trees of the communications graph. For reasons of this application, if we have a graph with numbers assigned to its edges, the sum of the numbers on the edges of a spanning tree of G will be called the *cost* of the spanning tree.

96. Describe a method (or better, two methods different in at least one aspect) for finding a spanning tree of minimum cost in a graph whose edges are labelled with costs, the cost on an edge being the cost for including that edge in a spanning tree.

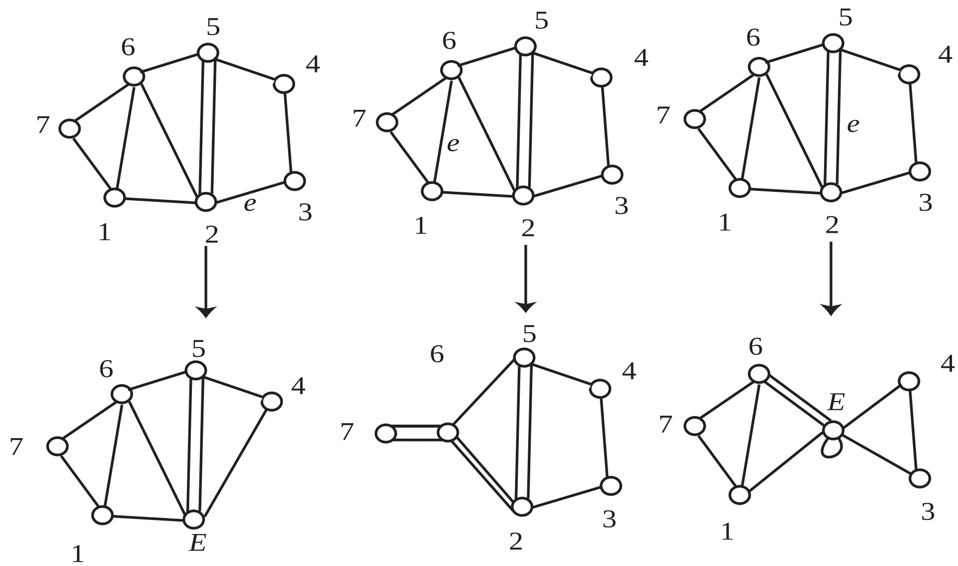
2.3.6 The deletion/contraction recurrence for spanning trees

There are two operations on graphs that we can apply to get a recurrence (though a more general kind than those we have studied for sequences) which will let us compute the number of spanning trees of a graph. The operations each apply to an edge e of a graph G. The first is called *deletion*; we *delete* the edge e from the graph by removing it from the edge set. The second operation is called *contraction*. Contractions of three different edges in the same graph are shown in Figure 2.4. We *contract* the edge e with endpoints v and w as follows:

1. remove all edges having either v or w or both as an endpoint from the edge set,
2. remove v and w from the vertex set,
3. add a new vertex E to the vertex set,
4. add an edge from E to each remaining vertex that used to be an endpoint of an edge whose other endpoint was v or w, and add an edge from E to E for any edge other than e whose endpoints were in the set $\{v, w\}$.

We use $G-e$ (read as G minus e) to stand for the result of deleting e from G, and we use G/E (read as G contract e) to stand for the result of contracting G from e.

Figure 2.4: The results of contracting three different edges in a graph.

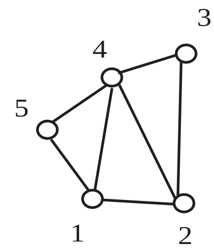


97. How do the number of spanning trees of G not containing the edge e and the number of spanning trees of G containing e relate to the number of spanning trees of $G-e$ and G/e? Use $\#(G)$ to stand for the number of spanning trees of G (so that, for example, $\#(G/e)$ stands for the number of spanning trees of G/e. Find an expression for $\#(G)$ in terms of $\#(G/e)$ and $\#(G-e)$. This expression is called the *deletion-contraction recurrence*. Use it to compute the number of spanning trees of the graph in Figure 2.5.

2.3.7 Shortest paths in graphs

Suppose that a company has a main office in one city and regional offices in other cities. Most of the communication in the company is between the main office and the regional offices, so the company wants to find a spanning tree that minimizes not the total cost of all the edges, but rather the cost of communication between the main office and each of the regional offices. It is not clear that such a spanning tree even exists. This problem is a special case of the following. We have a connected graph with numbers assigned to

Figure 2.5: A graph.



its edges. (In this situation these numbers are often called weights.) The *(weighted) length* of a path in the graph is the sum of the weights of its edges. The *distance* between two vertices is the least (weighted) length of any path between the two vertices. Given a vertex v, we would like to know the distance between v and each other vertex, and we would like to know if there is a spanning tree in G such that the length of the path in the spanning tree from v to each vertex x is the distance from v to x in G.

98. Show that the following algorithm (known as Dijkstra's algorithm) applied to a weighted graph whose vertices are labelled 1 to n gives, for each i, the distance from vertex 1 to v as $d(i)$.

    (a) Let $d(1) = 0$. Let $d(i) = \infty$ for all other i. Let $v(1)$=1. Let $v(j) = 0$ for all other j. For each i and j, let $w(i,j)$ be the minimum weight of an edge between i and j, or ∞ if there are no such edges. Let $k = 1$. Let $t = 1$.

    (b) For each i, if $d(i) > d(k) + w(k,i)$ let $d(i) = d(k) + w(k,i)$.

    (c) Among those i with $v(i) = 0$, choose one with $d(i)$ a minimum, and let $k = i$. Increase t by 1.

    (d) Repeat the previous steps until $t = n$

99. Is there a spanning tree such that the distance from vertex 1 to vertex i given by the algorithm in Problem 98 is the distance for vertex 1 to vertex i in the tree (using the same weights on the edges, of course)?

2.4 Supplementary Problems

1. A hydrocarbon molecule is a molecule whose only atoms are either carbon atoms or hydrogen atoms. In a simple molecular model of a hydrocarbon, a carbon atom will bond to exactly four other atoms and hydrogen atom will bond to exactly one other atom. We represent a hydrocarbon compound with a graph whose vertices are labelled with C's and H's so that each C vertex has degree four and each H vertex has degree one. A hydrocarbon is called an "alkane" (common examples are methane (natural gas), propane, hexane (ordinary gasoline), octane (to make gasoline burn more slowly), etc.) if each C vertex is adjacent to four distinct vertices and the graph is a tree. How many different alkanes have exactly n vertices labelled C? (Here we say two trees are the same if we can make their drawings congruent by shortening and lengthening lines, or moving the vertices and edges around, making sure that after we move things around, the edges are attached to the same vertices as before.)

2. (a) Give a recurrence for the number of ways to divide $2n$ people into sets of two for tennis games. (Don't worry about who serves first.)

   (b) Give a recurrence for the number of ways to divide $4n$ people into sets of four for games of bridge. (Don't worry about how they sit around the bridge table or who is the first dealer.)

3. Use induction to prove your result in Supplementary Problem 2 at the end of Chapter 1.