WRITTEN HW #3, DUE OCT 17 2011

Remember to write clearly and to justify all your claims in your solutions. Please staple your assignment before turning it in.

(1) (10 points) Suppose that $\gcd(q, a) = 1$. Dirichlet's Theorem (which we stated but never proved) says that there are infinitely many primes of the form $qk+a$, where $k \in \mathbb{Z}$. On the other hand, show that there are infinitely many values of k such that $qk + a > 0$ and $qk + a$ is composite.

(2) (10 points) Recall that we defined the binomial coefficient n choose m to equal

$$\binom{n}{m} = \frac{n!}{m!(n-m)!},$$

and that in the first homework assignment we saw this was equal to an integer. Let p be a prime, and let $0 < i < p$. Show that the power of p appearing in the factorization of $\binom{p}{i}$ is 1; ie, show that $p \,||\, \binom{p}{i}$.

(3) (10 points) Let p be a prime, and let n be a positive integer. Find an expression for the power of p in the factorization of $\mathrm{lcm}(1, 2, 3, \ldots, n)$, and prove that your answer is correct.

(4) (10 points) Let $a, b > 1$ be two integers which do not have all the same prime factors. (For instance, $a = 6, b = 24$ would not satisfy this property, since their prime factors are the same; namely, $2, 3$, whereas $a = 10, b = 8$ would, since $5 \mid a, 5 \nmid b$.) Show that $\log_a b$ is an irrational number.

(5) (10 points) Show that there are infinitely many prime numbers in the form $8k + 5$ or $8k + 7$.