HOMEWORK ASSIGNMENT #1, DUE FRIDAY, 10/1/2010

Remember to write clearly and to justify all your claims in your solutions.

(1) Use induction to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(2) Use the Euclidean algorithm to find the gcd of the following pairs of integers:
  (a) $a = 186, b = 51$,
  (b) $a = 438, b = 150$.

You should write down each step of the Euclidean algorithm (although you need not show all your arithmetic scratchwork).

(3) Use the Euclidean algorithm to find a pair of integer solutions x, y to the equation $96x + 28y = 8$. (Soon we will see how to find all integer solutions to this equation.)

(4) Suppose we know that the gcd of two positive integers, say a, b, is equal to 20. Is it possible to determine what all the (positive) common divisors of a, b are from this information? If so, what are those common divisors? (Remember, you need to prove all your assertions!)

(5) Recall that the *Fibonacci sequence* f_n is defined by the recursive relation $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$, and the initial terms $f_1 = f_2 = 1$. So the first few members of the Fibonacci sequence are $1, 1, 2, 3, 5, 8, \ldots$. Show that $\gcd(f_{n+1}, f_n) = 1$ for all $n \geq 1$.

(6) For each of the following sets of integers, determine whether they are mutually coprime, not mutually coprime but coprime, or not coprime. (Remember, prove your answer.)
  (a) $27, 80, 13$,
  (b) $24, 19, 186$.

(7) For a positive integer n, let $\phi(n)$ be the number of positive integers less than or equal to n which are coprime to n. For instance, $\phi(4) = 2$, since $1, 3$ are coprime to 4, but $2, 4$ are not, while $\phi(6) = 2$ as well, because $1, 5$ are coprime to 6, but $2, 3, 4, 6$ are not. (This function is called the *Euler totient* function and we will learn much more about it later in the class.)
  (a) Calculate $\phi(3)$ and $\phi(12)$.
  (b) Calculate $\phi(5)$ and $\phi(15)$. What is the relationship between $\phi(5), \phi(3)$, and $\phi(15)$?
  (c) Based on the above calculations, if a, b are positive integers, what do you conjecture for the relationship between $\phi(a), \phi(b)$, and $\phi(ab)$? (Don't bother trying to prove your conjecture, we'll do this later on.)
  (d) Test your conjecture with $a = 4, b = 2$. Do you need to change your conjecture at all? (Presumably you will need to calculate $\phi(2)$ and $\phi(8)$.)