Math 25: Solutions to Homework #2

(2.3 # 6) Suppose that m is a positive real nubmer. Show that $\sum_{j=1}^{n} j^m$ is $O(n^{m+1})$.

We see that

$$\sum_{j=1}^{n} j^m \leq \sum_{j=1}^{n} n^m = n \cdot n^m = n^{m+1},$$

so $\sum_{j=1}^{n} j^m \leq \sum_{j=1}^{n} n^m$ is $O(n^{m+1})$.

(2.3 # 8) Show that if f_1 is $O(g_1)$ and f_2 is $O(g_2)$, and c_1 and c_2 are constants, then $c_1 f_1 + c_2 f_2$ is $O(g_1 + g_2)$.

Since f_1 is $O(g_1)$ and f_2 is $O(g_2)$, there are positive constants k_1 and k_2 such that $f_1 \leq k_1 g_1$ and $f_2 \leq k_2 g_2$. Let $K = \max\{|c_1 k_1|, |c_2 k_2|\}$. Then

$$c_1 f_2 + c_2 f_2 \leq c_1 k_1 g_1 + c_2 k_2 g_2 \leq K(g_1 + g_2),$$

so $c_1 f_1 + c_2 f_2$ is $O(g_1 + g_2)$.

(3.2 # 4) Find the smallest four sets of prime triplets of the form p, $p+4$, $p+6$.

$$(5, 7, 11), \quad (11, 13, 17), \quad (17, 19, 23), \quad \text{and} \quad (41, 43, 47).$$

(3.2 # 12) Show that every integer greater than 11 is the sum of two composite integers.

Let $n \geq 12$. If n is even, there is an integer $k \geq 6$ such that $n = 2k$. Then $n = 2(k-2)+4$. Since $k - 2 > 1$, the two numbers in the sum are both composte. If n is odd, there is an integer $m \geq 6$ such that $n = 2m+1$. Then $n = 2(m-4)+9$. Since $m-4 \geq 2$, both numbers in the sum are composite.

(3.2 # 14(a)) Find $G(n)$ for all even integers n with $4 \leq n \leq 30$.

n	G(n)	Sums
4	1	2+2
6	1	3+3
8	1	3+5
10	2	5+5, 3+7
12	1	5+7
14	2	7+7, 3+11
16	2	3+13, 5+11
18	2	7+11, 5+13
20	2	7+13, 3+17
22	3	11+11, 5+17, 3+19
24	3	11+13, 7+17, 5+19
26	3	13+13, 7+19, 3+23
28	2	11+17, 5+23
30	3	13+17, 11+19, 7+23

(3.3 # 8) Show that if a and b are integers with $(a,b) = 1$, then $(a+b, a-b) = 1$ or 2.

Suppose that d is a common divsor of $a+b$ and $a-b$. Then $d|(a+b+a-b) = 2a$, and $d|((a+b)-(a-b)) = 2b$, so $d|(2a, 2b)$. Since $(a,b) = 1$, $(2a, 2b) = 2$, so $d|2$. Then $d = 1$ or $d = 2$. Therefore, the greatest common divisor of of $a+b$ and $a-b$ must also be either 1 or 2.

(3.3 # 12) Show that if a, b, and c are integers such that $(a,b) = 1$ and $c \mid (a+b)$, then $(c,a) = (c,b) = 1$.

Since $(a,b) = 1$, we can write $ma+nb = 1$ for some integers m and n. Then since $c|(a+b)$, there is an integer d such that $cd = a+b$. Then $b = cd - a$, so substituting in the linear combination, we have $ma + n(cd-a) = 1$, so $a(m-n) + c(nd) = 1$, and hence $(a,c) = 1$. Making the substitution $a = cd - b$, we have $m(cd-b) + nb = 1$, so $b(n-m) + c(md) = 1$, therefore $(b,c) = 1$ as well.

(3.3 # 16) Find four integers that are mutually relatively prime, but any two of which are not relatively prime.

There are many possible answers. One is $105 = 3 \cdot 5 \cdot 7$, $70 = 2 \cdot 5 \cdot 7$, $42 = 2 \cdot 3 \cdot 7$, $30 = 2 \cdot 3 \cdot 5$.

(3.4 # 2) Use the Euclidean algorithm to find each of the following GCDs.

(b) Find $(105, 300)$.

$$\begin{aligned} 300 &= 2 \cdot 105 + 90 \\ 105 &= 1 \cdot 90 + 15 \\ 90 &= 6 \cdot 15. \end{aligned}$$

So $(105, 300) = 15$.

(c) Find $(981, 1234)$.

$$\begin{aligned} 1234 &= 1 \cdot 981 + 253 \\ 981 &= 3 \cdot 253 + 222 \\ 253 &= 1 \cdot 222 + 31 \\ 222 &= 7 \cdot 31 + 5 \\ 31 &= 6 \cdot 5 + 1 \\ 5 &= 5 \cdot 1. \end{aligned}$$

So $(981, 1234) = 1$.

(3.4 # 4) Express the GCDs above as a linear combination of the integers.

Working backwards from the Euclidean Algorithm in the previous problem:

(b) $15 = 105 - 90 = 105 - (300 - 2 \cdot 105) = 3 \cdot 105 - 1 \cdot 300$.

(c)

$$\begin{aligned} 1 &= 31 - 6 \cdot 5 \\ &= 31 - 6(222 - 7 \cdot 31) = 43 \cdot 31 - 6 \cdot 222 \\ &= 43(253 - 222) - 6 \cdot 222 = 43 \cdot 253 - 49 \cdot 222 \\ &= 43 \cdot 253 - 49(981 - 3 \cdot 253) = 190 \cdot 253 - 49 \cdot 981 \\ &= 190(1234 - 981) - 49 \cdot 981 = 190 \cdot 1234 - 239 \cdot 981. \end{aligned}$$

(3.5 # 14) Let n be a positive integer. Show that the power of the prime p occurring in the prime-power factorization of $n!$ is

$$[n/p] + [n/p^2] + [n/p^3] + \cdots .$$

Since n is the product of the integers $1, 2, \ldots, n$, the prime factorization of $n!$ is the product of the prime factorizations of the integers $1, 2, \ldots, n$. To count the factors of a given prime p, we note that each multiple of p less than or equal to n contributes one p, and there are $[n/p]$ such numbers. Now every multiple of p^2 contributes 2 factors of p, but we have already counted one of the p's for each of these numbers, since they are also multiples of p. So we need only count again the number of multiples of p^2 up to n, of which there are $[n/p^2]$. Continuing in this manner, we see that there are

$$[n/p] + [n/p^2] + [n/p^3] + \cdots$$

factors of p in $n!$. Note that although this looks like an infinite sum, it only has finitely many nonzero terms, since for a fixed n, we will eventually come to a power $p^k > n$. Then $[n/p^j] = 0$ for all $j \geq k$.

(3.5 # 16) How many zeros are there at the end of 1000!?

The number of zeros at the end of any decimal expansion of a number corresponds exactly to the number of factors of 10 that divide the number. Since $10 = 2 \cdot 5$, and there are more multiples of 2 than of 5 among the integers $1, 2, \ldots, n$, we can simply count the number of factors of 5 in 1000!. Using the formula from the previous problem, we have

$$[1000/5] + [1000/25] + [1000/125] + [1000/625] = 200 + 40 + 8 + 1 = 249.$$

(3.5 # 32(d)) Find the GCD and LCM of $41^{101}47^{43}103^{1001}$ and $41^{11}43^{47}83^{111}$.

The GCD is 41^{11}, and the LCM is $41^{101}43^{47}47^{43}83^{111}103^{1001}$.

(3.5 # 54) Let n be a positive integer. How many pairs of positive integers satisfy $[a, b] = n$?

Let $n = p_1^{c_1} p_2^{c_2} \cdot p_t^{c_t}$. If $n = [a, b]$, then we can write $a = p_1^{a_1} p_2^{a_2} \cdot p_t^{a_t}$ and $b = p_1^{b_1} p_2^{b_2} \cdot p_t^{b_t}$ where for $1 \leq i \leq t$, $c_i = \max\{a_i, b_i\}$. To count the possible pairs a, b, we first note that there are 2^t ways to choose the primes p_i for which $p_i^{c_i} | a$. Then for each prime p_i, there are $c_i + 1$ choices for the power of p_i dividing whichever of a or b does not receive the maximum power of p_i (since then the power can be any number $0, 1, 2, \ldots, c_i$). So we have

$$2^t \prod_{i=1}^{t} (c_i + 1)$$

pairs a and b. But we have actually counted each pair twice, since we have counted (a, b) as distinct from (b, a). Since these are really the same pair of two numbers, and the order does not matter when calculating LCMs, we should divide by 2. So there are

$$2^{t-1} \prod_{i=1}^{t} (c_i + 1)$$

pairs a and b such that $n = [a, b]$.