Math 25: Solutions to Homework #1

(1.3 # 8) Use mathematical induction to prove that $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$ for every integer n.

We use the first principle of mathematical induction. For the base case, $1^3 = 1 = \left(\frac{1\cdot 2}{2}\right)^2$. For the induction hypothesis, assume that $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$ for some positive integer n. Then

$$\sum_{j=1}^{n+1} j^3 = (n+1)^3 + \sum_{j=1}^{n} j^3 = (n+1)^3 + \left(\frac{n(n+1)}{2}\right)^2 = \frac{4(n+1)^3 + n^2(n+1)^2}{4}$$
$$= \frac{(n+1)^2(n^2+4n+4)}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

Therefore, $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$ for every integer n.

(1.3 # 14) Show that any amount of postage that is an integer number of cents greater than 53 cents can be formed using just 7-cent and 10-cent stamps.

We use the second principle of mathematical induction. First note that n cents can be formed using 7-cent and 10-cent stamps if there are integers a and b such that $7a+10b = n$. For the base cases, we see that $54 = 2\cdot 7+4\cdot 10$, $55 = 5\cdot 7+2\cdot 10$, $56 = 8\cdot 7$, $57 = 1\cdot 7+5\cdot 10$, $58 = 4\cdot 7+3\cdot 10$, and $59 = 7\cdot 7+1\cdot 10$. For the induction hypothesis, assume that there is an integer solution to the equation $53+k = 7a+10b$ for $k = 1, 2, \ldots, n$. Then $53+n+1 = 53+(n-6)+7$, and by the induction hypothesis, there are integers a and b such that $53+(n-6) = 7a+10b$, so $53+n+1 = 7(a+1)+10b$. Therefore any number of cents greater than 53 can be formed using 7 and 10 cent stamps.

(1.3 # 30) Show that $2^n > n^2$ whenever n is an integer greater than 4.

We proceed by induction on n. For the base case, $2^5 = 32 > 5^2$. For the induction hypothesis, assume that $2^n > n^2$ some $n > 4$. Then $2^{n+1} = 2\cdot 2^n > 2n^2$ by the induction hypothesis. We need to show that $2n^2 > (n+1)^2 = n^2+2n+1$, so it is sufficient to show that $n^2 > 2n+1$. Consider the polynomial x^2-2x-1. By the quadratic formula, the two roots of this polynomial are $1 \pm \sqrt{2}$. Since the function $f(x) = x^2-2x-1$ is positive for $x > 1+\sqrt{2}$, and $n > 4 > 1+\sqrt{2}$, we see that $n^2 > 2n+1$. Therefore $2^n > n^2$ for all $n > 4$.

(1.4 # 6) Prove that $f_{n-2} + f_{n+2} = 3f_n$ whenever n is an integer with $n \geq 2$.

We use the second principle of mathematical induction. For the base case, we see that $f_0 + f_4 = 0 + 3 = 3f_2$. For the induction hypothesis, assume that $f_{k-2} + f_{k+2} = 3f_k$ for

$k = 2, 3, \ldots, n$. Then by the definition of the Fibonacci sequence,

$f_{n-1}+f_{n+3} = f_{n-2}+f_{n-3}+f_{n+2}+f_{n+1} = (f_{n-2}+f_{n+2})+(f_{n-3}+f_{n+1}) = 3f_n+3f_{n-1} = 3f_{n+1}$. Therefore $f_{n-2} + f_{n+2} = 3f_n$ whenever n is an integer with $n \geq 2$.

(1.5 # 12) Show that the sum of two even or of two odd integers is even, whereas the sum of an odd and an even integer is odd.

Let a and b be even integers. Then $a = 2k$ and $b = 2\ell$ for some integers k and ℓ. So $a + b = 2k + 2\ell = 2(k + \ell)$, and hence $a + b$ is even. Now suppose that a and b are both odd. Then $a = 2k + 1$ and $b = 2\ell + 1$ for integers k and ℓ. Then $a + b = 2k + 1 + 2\ell + 1 = 2(k+\ell)+2 = 2(k+\ell+1)$, so $a+b$ is even. Finally, suppose that a is even and b is odd. Then we write $a = 2k$ and $b = 2\ell+1$ for integers k and ℓ. Then $a+b = 2k+2\ell+1 = 2(k+\ell)+1$, and hence $a + b$ is odd.

(1.5 # 14) Show that if a and b are odd positive integers and $b \nmid a$, then there are integers s and t such that $a = bs + t$, where t is odd and $|t| < b$.

By the Division Algorithm, since $b \nmid a$, there are unique integers q and r with $0 < r < b$ such that $a = bq + r$. If q is even, then bq is even, so $r = a - bq$ is odd, and clearly $|r| < b$. If q is odd, then $a = b(q + 1) + (r - b)$. Then $q + 1$ is even, so $b(q + 1)$ is even, and hence $r - b = a - b(q + 1)$ is odd. Since $0 < r < b$, we see that $-b < r - b < 0$, so $|r - b| < b$. Therefore in each case, we have written $a = bs + t$, where t is odd and $|t| < b$.

(3.1 # 6) Show that no integer of the form $n^3 + 1$ is a prime, other than $2 = 1^3 + 1$.

Let n be an integer. Then $n^3 + 1 = (n + 1)(n^2 - n + 1)$, so $n^3 + 1$ is composite unless either $n + 1 = 1$ or $n + 1 = n^3 + 1$. If $n + 1 = 1$, then $n = 0$, so $n^3 + 1 = 1$, which is not prime. If $n+1 = n^3+1$, then $n = n^3$, and hence $n = 1$, so $n^3+1 = 2$. This is the only case in which $n^3 + 1$ is prime.

(3.1 # 8) Show that the integer $Q_n = n! + 1$, where n is a positive integer, has a prime divisor greater than n. Conclude that there are infinitely many primes.

Fix an integer $n \geq 1$, and suppose that all primes p satisfy $p \leq n$. Let $Q_n = n! + 1$. Then $Q_n > 1$, so it has some prime divisor $q \leq n$, and hence $q|n!$. But then $q|(Q_n - n!) = 1$, which is a contradiction. So there must be a prime larger than n. Since n was arbitrary, there are infinitely many primes.

(3.1 # 24) Find all lucky numbers less than 100.

The lucky numbers are 1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, 67, 69, 73, 75, 79, 87, 93, 99.