Math 24, Winter 2020, Pset 5

This problem set is due on Friday March 6.

1. Prove that if $T : V \to V$ is a linear transformation and λ is an eigenvalue of T, then λ^2 is an eigenvalue of T^2.

2. Prove that if T is invertible, and λ is an eigenvalue of T, then λ^{-1} is an eigenvalue of T^{-1}.

3. Prove that a linear transformation $T : V \to V$ is invertible if and only if $\lambda = 0$ is *not* an eigenvalue of T.

4. Let $V = P_2(\mathbb{R})$. For each of the following linear transformations $T : V \to V$, test whether T is diagonalizable, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

   (a) T is defined by $T(f) = f + f'$.

   (b) T is defined by $T(ax^2 + bx + c) = cx^2 + bx + a$.

5. Let $L_A : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation associated to the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

   Here $\mathbb{C}^2$ is a vector space over the field $F = \mathbb{C}$.

   (a) Determine the eigenvalues of A.

   (b) For each eigenvalue, find the eigenspace.

   (c) Find a basis for $\mathbb{C}^2$ consisting of eigenvectors of A.

   (d) Determine an invertible matrix Q and a diagonal matrix D such that $A = Q^{-1}AQ$.

6. On the midterm exam, you calculated the 3×3 matrix $[UT]_\beta$ of the composition of the spatial rotations $T : \mathbb{R}^3 \to \mathbb{R}^3$ (about the z-axis by an angle of $\pi/3$) and $U : \mathbb{R}^3 \to \mathbb{R}^3$ (about the x-axis by angle $\pi/3$). In the standard basis $\beta = \{e_1, e_2, e_3\}$:

$$[UT]_\beta = [U]_\beta[T]_\beta = \begin{pmatrix} c & -s & 0 \\ cs & c^2 & -s \\ s^2 & cs & c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}$$

   with $c = \cos \pi/3 = \frac{1}{2}$, $s = \sin \pi/3 = \frac{\sqrt{3}}{2}$.

   In general, the composition of two spatial rotations is again a spatial rotation. In this problem you will calculate the axis and angle of rotation of UT.

   (a) Find the characteristic polynomial of UT.

   (b) Because $UT : \mathbb{R}^3 \to \mathbb{R}^3$ is a spatial rotation, $\lambda = 1$ must be one of its eigenvalues. Find the eigenspace for eigenvalue $\lambda = 1$.

(c) Factor the characteristic polynomial of UT.

(d) Is UT diagonalizable? Explain your answer.

(e) Find the complex roots of the characteristic polynomial.

(f) What is the axis of rotation of UT?

(g) What is the angle of rotation of UT?