Math 24, Winter 2020, Pset 2

This problem set is due at the start of lecture on Wednesday January 22.

1. In each case below, determine whether $v \in \text{span}(S)$.

   (a) In the vector space $V = \mathbb{R}^3$ (over $F = \mathbb{R}$), with

   $$v = (-1, 5, 5) \qquad S = \{(3, -1, 1), (1, 2, 3)\}$$

   (b) In the vector space $V = \mathbb{C}^3$ (over field $F = \mathbb{C}$), with

   $$v = (1, 0, 0) \qquad S = \{(i, 1, 0), (-i, 1, 0)\}$$

   (c) In the vector space $P_3(\mathbb{R})$ of polynomials of degree at most 3, with

   $$v = x^3 - 2x^2 \qquad S = \{x^3 + 2x + 2, x^2 + x + 3\}$$

2. Let F be a field, and $V = F^n$. Let e_j be the n-tupel where all components are 0, except the j-th component which is 1. So $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0)$ etc.

   (a) Show that $\{e_1, e_2, \ldots, e_n\}$ generates F^n.

   (b) Show that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

   (c) What is the dimension of F^n?

   (d) If $F = \mathbb{F}_5$ is the field with 5 elements, prove that $\{(1, 1), (2, 3)\}$ is a basis of F^2.

3. Let x, y, z be vectors in a vector space V over an arbitrary field F. Prove that:

   (a) If $\{x, y, z\}$ is linearly dependent then $\{x, x + y, x + y + z\}$ is linearly dependent.

   (b) If $\{x, x + y, x + y + z\}$ is a basis of V then $\{x, y, z\}$ is a basis of V.

4. From the theory of differential equations, it is known that all solutions of the equation $f''(x) + f(x) = 0$ are of the form $f(x) = a \cos x + b \sin x$ for arbitrary $a, b \in \mathbb{R}$.

   (a) Show that $\{\cos x, \sin x\}$ is a linearly independent subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

   (b) What is the dimension of the subspace $W = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f''(x) + f(x) = 0\}$?

   Remark. W is called the *solution space* of the equation $f''(x) + f(x) = 0$.

5. Prove that $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $\mathbb{R}^3$.

6. Let W_1 and W_2 be two finite dimensional subspaces of a vector space V. Suppose that $W_1 \cap W_2 = \{0\}$, while the union $W_1 \cup W_2$ generates V. Prove that

   $$\dim(V) = \dim(W_1) + \dim(W_2)$$

   Hint. Show that the union of a basis for W_1 and a basis for W_2 is a basis for V.