Math 24

Winter 2017

Special Assignment due Monday, February 20

Let V be any vector space over F and W be a subspace of V. For any vector x in V, we defined the *coset* of W containing x to be

$$x + W = \{x + w \mid w \in W\}.$$

We denote the collection of cosets by V/W.

It turns out that V/W forms a vector space over F, with operations defined by

$(x + W) + (y + W) = (x + y) + W$

$a(x + W) = (ax) + W$.

You may assume that this is true. (You proved part of this in the last two special homework assignments.)

Assignment: We can define a function T from V to V/W by $T(x) = x + W$.

Prove that T is a linear transformation.

Identify the null space and range of T.

If V is finite-dimensional, what can you conclude about the dimensions of V, W, and V/W?

General comments: It is important to keep straight the difference between vectors in V and cosets in V/W.

The domain of T is V, and so $N(T)$ is a subset of V. Its elements are vectors in V. To identify the null space of T, you must find which vectors $v \in V$ satisfy $T(v) = 0_{V/W}$. Remember that in the last assignment you showed that $0_{V/W} = 0_V + W$.

The codomain of T is V/W, and so $R(T)$ is a subset of V/W. Its elements are cosets. To identify the range of T, you must find which cosets $x + W \in V/W$ satisfy $x + W = T(v)$ for some vector $v \in V$.

As a concrete example, consider the example in which $V = \mathbb{R}^2$, and W is the x-axis. We saw that the elements of V/W are horizontal lines (lines given by equations $y = b$), and for a vector $(a, b) \in V$, the coset of (a, b) is the line $y = b$. Then $T((a, b))$ is the line with equation $y = b$.

The zero vector of V/W is the coset $\vec{0} + W$. In this example, $0_{V/W}$ is the coset $(0, 0) + W$, or the line $y = 0$; that is, the x-axis.

The null space of T consists of all vectors $(a, b) \in \mathbb{R}^2$ for which $T((a, b)) = \vec{0}_{V/W}$; that is, all vectors $(a, b) \in \mathbb{R}^2$ for which the line with equation $y = b$ is the x-axis; or, all vectors $(a, b) \in \mathbb{R}^2$ for which $b = 0$. This is the x-axis, or W. In particular, notice that the elements of $N(T)$ are elements of R^2, or points.

The range of T consists of all horizontal lines that are $T((a, b))$ for some vector $(a, b) \in \mathbb{R}^2$; that is, all horizontal lines that have equation $y = b$ for some vector $(a, b) \in \mathbb{R}^2$. This includes all horizontal lines, so the range of T is the entire space V/W. In particular, notice that the elements of $R(T)$ are horizontal lines, not points.

In this case, the dimension of $N(T)$ is 1, and the dimension of $R(T)$ is also 1.

To see that the dimension of $R(T)$ is 1, note that the line $y = b$ is the coset

$$(0, b) + W = (b(0, 1)) + W = b\big((0, 1) + W\big).$$

The last step is because scalar multiplication in V/W is defined by $b(x + W) = (bx) + W$. This shows that every coset is a multiple of the coset $(0, 1) + W$, so the single element $(0, 1) + W$ forms a basis for V/W.

Intuitively, you may at first think that because the cosets cover the entire two-dimensional space $\mathbb{R}^2$, the dimension of V/W should be 2. However, each coset is itself a line, which is a one-dimensional object. Very loosely, we have collapsed a one-dimensional piece of $\mathbb{R}^2$ into a single element of V/W, which is why we lose a dimension.