Math 24
Winter 2017
Special Assignment due Monday, February 6

Let V be any vector space and W be a subspace of V. For any vector x in V, we define the *coset* of W containing x to be

$$x + W = \{x + w \mid w \in W\}.$$

We denote the collection of cosets of W in V by V/W:

$$V/W = \{x + W \mid x \in V\}.$$

For your last assignment, you proved that addition of cosets is well-defined, where

$$(x + W) + (y + W) = (x + y) + W.$$

Assignment: Prove that V/W, with addition defined as above, satisfies vector space axioms (VS2), (VS3), and (VS4).

Note that for (VS3), for example, you should choose a specific element of V/W, and show that element is an additive identity.

As an example, here is a proof that V/W satisfies axiom (VS1). It is followed by a proof that, for vector spaces V and Z, the space $V \times Z$ (as defined in the first midterm) satisfies axiom (VS3). This should have been an ingredient in your answer to the first problem on the midterm.

Proposition: Let W be a subspace of a vector space V. Addition of cosets in V/W is commutative.

Proof: Let $X, Y \in V/W$. Then $X = x + W$ and $Y = y + W$ for some $x, y \in V$. By the definition of addition of cosets, we have

$$\begin{aligned} X + Y &= (x + W) + (y + W) \\ &= (x + y) + W \\ &= (y + x) + W \\ &= (y + W) + (x + W) \\ &= Y + X. \end{aligned}$$

This is what we needed to prove.

Note: In going from $(x+y)+W$ to $(y+x)+W$, I used the fact that addition of vectors in V is commutative without comment. At this point in the course, you can assume your reader is familiar with the vector space axioms. However, in the context of a particular proof, it may make things easier to follow if you point out where you are using the vector space axioms. Use your judgment.

Proposition: If V and Z are vector spaces, then the space $V \times Z$ (as defined in the first midterm) satisfies axiom (VS3).

Proof: We need to show that $V \times Z$ has an additive identity. Let 0_V and 0_Z be the additive identity elements of V and Z, respectively. We will show that $(0_V, 0_Z)$ is an additive identity for $V \times Z$.

To show this, let (v, z) be any element of $V \times Z$. We must show $(v, z) + (0_V, 0_Z) = (v, z)$. Using the definition of addition in $V \times Z$, we have

$$(v, z) + (0_V, 0_Z) = (v + 0_V,\ z + 0_Z) = (v, z).$$

QED.

Note: (This note is just cultural enrichment. You can ignore it, or read it later.) We can make a similar definition for other sorts of structures and substructures. For example, the integers $\mathbb{Z}$ with addition and multiplication form a "commutative ring with unity." This is a structure that satisfies all the axioms for a field except possibly the existence of multiplicative inverses. The set of multiplies of n

$$n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$$

is a kind of substructure of $\mathbb{Z}$ called an "ideal." This means it is closed under addition, and also under multiplication by any element of $\mathbb{Z}$. Now if we define cosets of $n\mathbb{Z}$ the same way we did above,

$$x + n\mathbb{Z} = \{x + m \mid m \in n\mathbb{Z}\},$$

we can define addition and multiplication of cosets

$$(x + n\mathbb{Z}) + (y + n\mathbb{Z}) = (x + y) + n\mathbb{Z} \quad \text{and} \quad (x + n\mathbb{Z})(y + n\mathbb{Z}) = (xy) + n\mathbb{Z}.$$

We get the structure $\mathbb{Z}/n\mathbb{Z}$, whose elements are cosets $0 + n\mathbb{Z}$, $1 + n\mathbb{Z}$, $\ldots (n-1) + n\mathbb{Z}$.

$\mathbb{Z}/2\mathbb{Z}$ is the same as $\mathbb{Z}_2$ (defined in Appendix C of the textbook), except that instead of calling the elements $0 + 2\mathbb{Z}$ and $1 + 2\mathbb{Z}$, the textbook just calls them 0 and 1. Another name for $\mathbb{Z}/n\mathbb{Z}$ is "the integers modulo n." If you're up for a challenge, you might notice that while $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are fields, $\mathbb{Z}/4\mathbb{Z}$ is not. For which n is $\mathbb{Z}/n\mathbb{Z}$ a field?