Math 24

Winter 2017

Special Assignment due Monday, January 30

Be warned, there are a lot of words in this document. You should read them all. However, the assignment is given in a box on the next page.

Let V be any vector space and W be a subspace of V. For any vector x in V, we define the *coset* of W containing x to be $x + W = \{x + w \mid w \in W\}$.

For your last assignment, you proved that for any vector space V and any subspace W of V,

1. For any x in V, $x \in (x + W)$;
2. For any x and y in V, $(x - y) \in W \implies (x + W) = (y + W)$;
3. For any x and y in V, $(x - y) \notin W \implies (x + W) \cap (y + W) = \emptyset$.

That is, the cosets of W form a *partition* of V; different (unequal) cosets are disjoint (do not overlap), and together, the cosets cover all of V. In the example given in the assignment, $V = \mathbb{R}^2$ and W is the x-axis, and the cosets of W are all the lines parallel to the x-axis: $(a, b) + W$ is the line $y = b$.

Now we will define a method of adding cosets to each other.

Definition: If $X = x + W$ and $Y = y + W$ are two cosets of W, we define their sum to be

$$X + Y = (x + y) + W.$$

We could also say simply,

$$(x + W) + (y + W) = (x + y) + W.$$

In the given example, $(1, 2) + W$ is the line $y = 2$, and $(1, -1) + W$ is the line $y = -1$. Then by our definition,

$$\big((1, 2) + W\big) + \big((1, -1) + W\big) = \big((1, 2) + (1, -1)\big) + W = (2, 1) + W,$$

which is the line $y = 1$.

For this definition to make sense, we have to know that if, for example, we think of the line $y = 2$ as the coset $(3, 2) + W$, and the line $y = -1$ as the coset $(-5, -1) + W$, and use this definition to add those two lines, we get the same answer. It's easy to check in this specific case; our definition would say

$$\big((3, 2) + W\big) + \big((-5, -1) + W\big) = \big((3, 2) + (-5, -1)\big) + W = (-2, 1) + W,$$

which is indeed the line $y = 1$.

We need to know that this always happens. If it does we say that addition of cosets is *well-defined*.

Assignment: Show that addition of cosets is well-defined.

That is, show that whenever we have

$$X = x + W = x' + W \text{ and } Y = y + W = y' + W$$

then we have

$$(x + y) + W = (x' + y') + W.$$

(Only then do we know that it makes sense to talk about $X + Y$.)

To begin this proof, you may be tempted to say the following.

> Suppose that $x + W = x' + W$ and $y + W = y' + W$.
> We must show that $(x + W) + (y + W) = (x' + W) + (y' + W)$.

DO NOT SAY THAT. You should not use the expression $(x + W) + (y + W)$ until you know that addition of cosets makes sense. If you can come up with two different answers for $X + Y$ depending on how you write X and Y, then addition of cosets does not make sense. You are still trying to prove that can't happen. **SAY THIS:**

> Suppose that $x + W = x' + W$ and $y + W = y' + W$.
> We must show that $(x + y) + W = (x' + y') + W$.

You have permission to use these exact words to begin your proof. If you do that, you will be heading in the right direction.

Here is an example of a proposed operation that is *not* well-defined. For subspaces W of $\mathbb{R}^n$ (where we have the notion of dot product), suppose we try to define the dot product of two cosets by

$$(x + W) \cdot (y + W) = (x \cdot y).$$

Then, in the same example as before, if we take $X = (1, 2) + W = (3, 2) + W$ and we take $Y = (1, -1) + W = (-5, -1) + W$, then we have

$$(1, 2) \cdot (1, -1) = -1 \text{ and } (3, 2) \cdot (-5, -1) = -17,$$

so $X \cdot Y$ would have to equal both -1 and -17. That means this notion of the dot product of two cosets is NOT well-defined, and we can't use this definition.

The moral: When you are defining some function of X, if your definition depends on how you name or express X, and if there can be more than one way to name or express the same X, then you must verify that your function is well-defined. If it is not, then you have not actually defined a function.

The same thing applies if there is some other reason that your definition might assign more than one value to some X. For example, if we tried to define a function $f : \mathbb{C} \to \mathbb{C}$ by setting $f(x)$ to be the number y such that $y^2 = x$, we would not have a well-defined function. This is because (unless x is 0) there are not one but two complex numbers y such that $y^2 = x$. Since the definition does not identify one and only one value for $f(x)$, the function is not well-defined.

For this class, at least, NEVER use the expression "$f(X)$" in the proof that f is well-defined. Use the definition of f instead. For example, in the proof that addition of cosets is well-defined, we do not use the expression $(x+W)+(y+W)$; instead, we use the expression $(x+y)+W$.

Here is an example of a (partially completed) proof that an operation is well-defined.

Example: Define multiplication of cosets by scalars by

$$a(x+W) = ax + W.$$

Proposition: This operation is well-defined.

Proof: Let a be a scalar and x and y vectors such that $x+W = y+W$. We must show that

$$ax + W = ay + W.$$

We have shown that if $x - y \notin W$, then $x + W \cap y + W = \emptyset$.

In this case, $x + W \cap y + W = x + W \neq \emptyset$ ($x + W$ must be nonempty since we have shown $x \in x + W$), so we must have $x - y \in W$.

First, suppose $z \in ax + W$ and show $z \in ay + W$. Since $z \in ax + W$, for some $w \in W$ we have

$$z = ax + w = a(y + (x - y)) + w = ay + [(a(x - y)) + w].$$

Since W is a subspace containing both $x - y$ and w, we have $(a(x - y)) + w \in W$. This shows $z \in ay + W$.

Now, suppose $z \in ay + W$ ...

Note: This proof uses the straightforward way of showing two sets are equal, by showing every element of the first belongs to the second, and vice versa. There is a shorter way to show that $ax+W = ay+W$, using the facts we proved in the first special assignment, which we can easily show imply the following:

$$x + W = y + W \iff (x - y) \in W;$$

$$x + W \cap y + W = \emptyset \iff (x - y) \notin W.$$

With a little more work you could use those facts to prove that

$$x + W = y + W \iff y \in x + W.$$