Math 24
Winter 2017
Special Assignment due Wednesday, January 18

Please hand in this assignment on a different piece of paper from the regular homework. Do not staple it together with the regular homework. (Be sure your name is on all assignments.)

Let V be any vector space and W be a subspace of V. For any vector x in V, we define the *coset* of W containing x to be

$$x + W = \{x + w \mid w \in W\}.$$

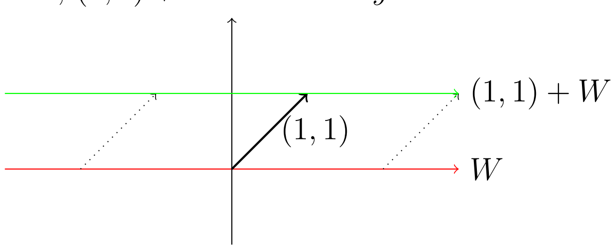
Example: If V is $\mathbb{R}^2$ or $\mathbb{R}^3$, you can picture $x + W$ geometrically as a translation of W by the vector x. For example, if $V = \mathbb{R}^2$ and W is the x-axis,

$$W = \{(x, 0) \mid x \in \mathbb{R}\},$$

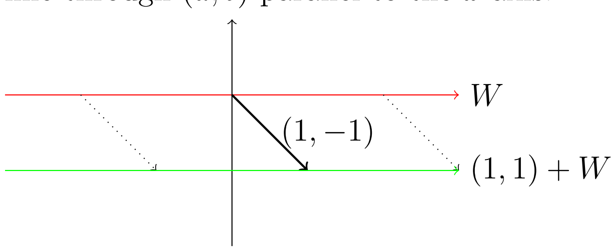
then the coset of W containing $(1, 1)$ is given by

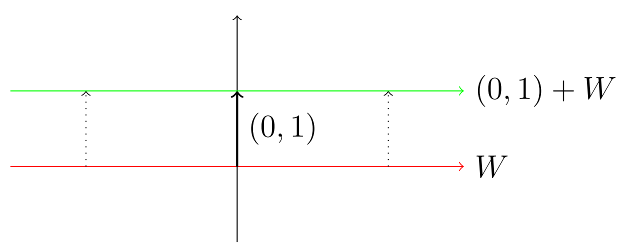
$$(1, 1) + W = \{(1, 1) + (x, y) \mid (x, y) \in W\} = \{(1, 1) + (x, 0) \mid x \in \mathbb{R}\} = \{(x + 1, 1) \mid x \in \mathbb{R}\}.$$

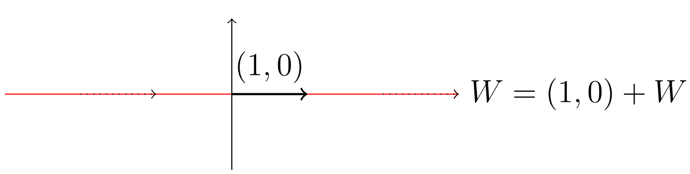
That is, $(1, 1) + W$ is the line $y = 1$.



In general, if W is the x-axis, then $(a, b) + W$, the coset of W containing (a, b), is the line through (a, b) parallel to the x-axis.







Example: In $P_2(\mathbb{Q})$, if

$$W = \{ax^2 + bx \mid a, b \in \mathbb{Q}\}$$

is the subspace of polynomials with constant term equal to 0, then the coset of $x + 2$ is

$$(x + 2) + W = \{(x + 2) + (ax^2 + bx) \mid a, b \in \mathbb{Q}\} = \{ax^2 + (b + 1)x + 2 \mid a, b \in \mathbb{Q}\},$$

the subset of polynomials with constant term equal to 2. (Don't be distracted by the fact that the coefficient of x is $(b + 1)$. Since b can be any rational number $(b + 1)$ can also be any rational number.)

Example: In $M_{2\times 2}(\mathbb{R})$, if W is the subspace of matrices $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ whose diagonal entries sum to 0, then $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + W$ is the subset of matrices $\begin{pmatrix} 1+a & b \\ 1+c & -a \end{pmatrix}$ whose diagonal entries sum to 1.

Assignment: Prove the following. For any vector space V and any subspace W of V:

1. For any x in V,
$$x \in (x + W).$$
2. For any x and y in V,
$$(x - y) \in W \implies (x + W) = (y + W).$$
3. For any x and y in V,
$$(x - y) \notin W \implies (x + W) \cap (y + W) = \emptyset.$$

Note: To show $x \in x + W$, you must show that $x = x + w$ for some $w \in W$.

If X and Y are sets, one way to prove $X = Y$ is to prove every element of X is in Y, and every element of Y is in X. One way to prove $X = \emptyset$ is to assume there is a element in X, and get a contradiction.

Note: You can check in the example on the first page that these conditions hold for the cosets pictured. For example, $(1, 1) - (0, 1)$ is $(1, 0)$, which belongs to W (the x-axis), and $(1, 1) + W$ and $(0, 1) + W$ are the same coset (the line $y = 1$).

Note: What you have shown is that the cosets of W form a *partition* of V. This means that different (unequal) cosets are disjoint (do not overlap), and together, the cosets cover all of V (every $x \in V$ belongs to some coset). In the example, the lines parallel to the x-axis form a partition of $\mathbb{R}^2$; every point belongs to exactly one line parallel to the x-axis.