Math 24
Winter 2017
Some Proof Principles

Generally, proving something requires some creativity; there is no recipe for producing a proof. However, there are some standard techniques that can be used, depending on the form of the statement you are trying to prove. (Note that "can" does not mean "must.") Here are a few of them.

1. To prove a statement of the form "If A, then B," assume A and prove B. Or, prove the *contrapositive*, "If not B, then not A," by assuming not B and proving not A.

2. To prove a statement of the form "A if and only if B" ($A \iff B$), prove "If A, then B," and prove, "If B, then A."

3. To prove a statement of the form "not A," use *proof by contradiction*: Assume A, and deduce a contradiction, something obviously false or contradictory.

4. To prove a statement of the form "For all vectors x, A(x)," let x be a name for an arbitrary vector, and prove A(x).

5. To prove a statement of the form "There is a vector x such that A(x)," find a specific example $\vec{v}$ and prove that A($\vec{v}$). (For example, prove that A($\vec{0}$).)

6. To prove a statement of the form "A and B," prove both A and B.

7. To prove a statement of the form "A or B," prove "If not A, then B," or prove "If not B, then A," or assume "Not A and not B" and deduce a contradiction. Or, consider all possible cases, and prove that in some cases A holds, and in other cases B holds.

8. In general, prove something by considering all possible cases separately. You must be sure the cases you list cover all possibilities. There is an example of a proof like this on page 4 of this handout.

9. To prove something is unique, assume there are two such things, and prove they are actually equal.

10. To prove a statement of the form "There is a unique x such that A(x)," prove both "There is an x such that A(x)" and "the x such that A(x) is unique." This is called proving existence and uniqueness.

Example: Prove that for every nonzero vector $\vec{v} \in \mathbb{R}^3$ there is a unique plane containing the origin that is normal to $\vec{v}$.

Proof: Let $\vec{v}$ be an arbitrary nonzero vector in $\mathbb{R}^3$.[1] We can write $\vec{v} = (a, b, c)$.

We know from Math 8 that the plane Π with equation

$$ax + by + cz = 0$$

has normal vector $(a, b, c) = \vec{v}$, and we can check (by plugging $x = y = z = 0$ into the equation) that it contains the origin. This proves existence.[2]

To prove uniqueness, suppose that the plane Σ with equation

$$\overline{a}x + \overline{b}y + \overline{c}z + \overline{d}$$

contains the origin and is perpendicular to $\vec{v}$. We must show that Σ equals Π.[3]

Since the origin is on Σ, by plugging $x = y = z = 0$ into the equation we can see that $\overline{d} = 0$. We can also see from the equation that a normal vector to Σ is $\vec{n} = (\overline{a}, \overline{b}, \overline{c})$. Since Σ is perpendicular to $\vec{v}$, its normal vector $\vec{n}$ must be parallel to $\vec{v}$, so $\vec{v}$ must be a scalar multiple of $\vec{n}$:

$$(a, b, c) = \vec{v} = s\vec{n} = (s\overline{a}, s\overline{b}, s\overline{c}).$$

Now we can rewrite the equation for Π,

$$ax + by + cz = 0,$$

as

$$s\overline{a}x + s\overline{b}y + s\overline{c}z = 0.$$

Since (a, b, c) is not the zero vector, $s \neq 0$, and we can rewrite the equation for Π by dividing by s. This gives

$$\overline{a}x + \overline{b}y + \overline{c}z = 0,$$

which is the equation for Σ. Therefore, Π equals Σ, which is what we needed to show.

[1] We are using (4) from the previous page. We are about to use (10) as well, proving existence and uniqueness.

[2] We are using (5) from the previous page, showing a particular plane has the given property.

[3] We are using something like (9) from the previous page, showing any other plane with the given property must equal the one we already found.

Writing Proofs

A mathematical proof of a statement is a clear, complete, and logically correct argument that the statement must be true. Here are a few important points about proofs:

1. Proofs are written in mathematical English. This means you should use complete sentences with correct grammar and punctuation.

2. You should use mathematical formulas, equations, and pictures in a proof, whenever they help make your proof readable and understandable.

3. Formulas, equations, and pictures should always be explained. A string of equations without explanations is not a proof.

4. Formulas and equations are included in sentences, and must be punctuated accordingly. Notice the punctuation in the following proof.

5. Always begin by stating the proposition you are going to prove.

6. Make the logic of your proof clear to your reader. If you are proving the additive identity of a vector space is unique, it is good to begin with, "Let $\vec{a}$ be an additive identity. We will prove that $\vec{a} = \vec{0}$."

7. How your proof is laid out on the paper matters. Centering equations on their own lines, and skipping lines between parts of a solution, can make your solution much more readable. Neatness counts.

8. It is fine to use formulas and results from the text or from class. Be sure you reader knows what axiom, formula, or result you are using.

9. There is generally more than one correct proof of a theorem, and more than one way to write up a given proof. Unless a homework or exam problem specifies a particular approach or technique, you can use any logically valid method of proof

10. The amount of detail needed in a proof depends on the intended reader. For this class, your intended reader should be a student in the class who does not understand the material quite as well as you do.

11. The mathematical "we" is common in proofs, but it is fine to use "I," as in, "Let $\vec{a}$ be an additive identity. I will prove that $\vec{a} = \vec{0}$

12. You are not required to use the notation $\vec{x}$ for vectors. However, it should always be clear when a symbol represents a vector and when it represents a scalar. ("Clear" does not mean "you can figure it out from context." It means *clear*.)

13. Professor Annalisa Crannell of Franklin and Marshall College has written a booklet about writing mathematics for her calculus classes. She discusses a number of strategies and conventions for writing mathematics well. You can find her booklet here:

    https://www.fandm.edu/uploads/files/107682389602454187-guide-to-writing.pdf

    Professor Steven Kleiman of MIT has written a more advanced guide to writing mathematics, intended for undergraduate students who are writing mathematical papers. You can find his guide here:

    http://www.mit.edu/afs/athena.mit.edu/course/other/mathp2/www/piil.html

14. Excellent mathematical writing style embodies several characteristics, of which the three most important are clarity, clarity, and clarity. It is important to use words precisely and correctly. Generally, simple declarative sentences and consistent word use are preferable to variation in sentence structure and vocabulary. The same is true of most technical writing; the deeper and more complex the ideas, the simpler and more transparent the writing should be. My favorite quotation about this comes from the web page "Guidelines for Writing a Phiilosophy Paper" by NYU philosophy professor James Pryor:[4]

    > If your paper sounds as if it were written for a third-grade audience, then you've probably achieved the right sort of clarity.

[4] http://www.jimpryor.net/teaching/guidelines/writing.html

Proposition: Suppose that $X \subseteq \mathbb{R}^2$ is a nonempty subset of $\mathbb{R}^2$ that is closed under addition and scalar multiplication. (This means that for every $\vec{x}$ and $\vec{y}$ in X, and every scalar a, the vectors $\vec{x} + \vec{y}$ and $a\vec{x}$ are also in X.) Then X must be one of:

1. $\{\vec{0}\}$ (the set whose only element is the zero vector);
2. a line through the origin;
3. all of $\mathbb{R}^2$.

Proof: There are three possible cases for X:

1. X contains no nonzero vectors;
2. X contains at least one nonzero vector, and all nonzero vectors in X are parallel;
3. X contains at least one pair of nonzero vectors that are not parallel.

We consider each case separately.

1. We will show that if X contains no nonzero vectors, then $X = \{\vec{0}\}$.

   X must contain at least one vector, since X is nonempty. Therefore, since X does not contain any nonzero vectors, X must contain the zero vector, and we have $X = \{\vec{0}\}$.

2. We will show that if X contains at least one nonzero vector, and all nonzero vectors in X are parallel, then X is a line through the origin.

   Let $\vec{v}$ be some nonzero element of X. If $\vec{w}$ is any other element of X, either $\vec{w} = \vec{0}$ or $\vec{w}$ is parallel to $\vec{v}$. In either case, $\vec{w}$ is a scalar multiple of $\vec{v}$; that is, $\vec{w} = t\vec{v}$ for some scalar t. This shows that every element of X is a scalar multiple of $\vec{v}$.

   Now, because X is closed under multiplication by scalars, *every* scalar multiple of $\vec{v}$ must be in X. Therefore X must consist exactly of all the scalar multiples of $\vec{v}$,

   $$X = \{t\vec{v} \mid t \in \mathbb{R}\}.$$

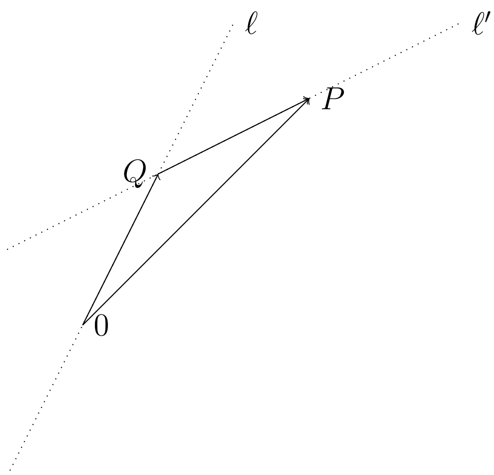
   That is, X is the line through the origin in the direction of $\vec{v}$.

3. We will show that if X contains contains at least one pair of nonzero vectors that are not parallel, then X is all of $\mathbb{R}^2$.

   Let $\vec{v}$ and $\vec{w}$ be nonzero, nonparallel elements of X. Because X is closed under both addition and multiplication by scalars, every vector of the form $s\vec{v} + t\vec{w}$ must be in X. To show $X = \mathbb{R}^2$, we must show every vector $(c_1, c_2) \in \mathbb{R}^2$ can be written in the form $s\vec{v} + t\vec{w}$.

   Method 1: Argue geometrically. Since $\vec{v}$ and $\vec{w}$ are not parallel, you can get from $(0, 0)$ to any point in the plane by proceeding some distance in the direction of $\vec{v}$ and then some distance in the direction of $\vec{w}$. That is, you can express any element of $\mathbb{R}^2$ as the sum of a scalar multiple of $\vec{v}$ and a scalar multiple of $\vec{w}$.

Specifically, let O be the origin, and P a point with coordinates (c_1, c_2); then $\overrightarrow{OP}$ is the vector (c_1, c_2). Let ℓ be the line through O parallel to $\vec{v}$ and ℓ' be the line through P parallel to $\vec{w}$. Since ℓ and ℓ' are not parallel, they intersect at some point Q, as in the picture.



By the geometric representation of vector addition,

$$\overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OP} = (c_1, c_2).$$

Because O and Q are both on a line ℓ parallel to $\vec{v}$, it follows that $\overrightarrow{OQ}$ is parallel to $\vec{v}$. That is, it is a scalar multiple of $\vec{v}$, so we can write $\overrightarrow{OQ} = s\vec{v}$. Similarly, since Q and P are both on ℓ', we can write $\overrightarrow{QP} = t\vec{w}$. Now we have

$$(c_1, c_2) = \overrightarrow{OQ} + \overrightarrow{QP} = s\vec{v} + t\vec{w}.$$

This is what we needed to show.

Method 2: Argue algebraically.

Suppose $\vec{v} = (a_1, a_2)$ and $\vec{w} = (b_1, b_2)$. We must show that for any choice of (c_1, c_2) we can find real numbers s and t such that

$$s(a_1, a_2) + t(b_1, b_2) = (c_1, c_2).$$

That is, we must show we can always solve the system of linear equations

$$a_1 s + b_1 t = c_1$$

$$a_2 s + b_2 t = c_2$$

for s and t.

(Note: I was able to come up with the following argument because I already know linear algebra. It uses Cramer's Rule, page 224 of the textbook. You might come up with a similar argument by trying to solve the system of linear equations, and seeing what you need to assume in order to solve it.)

We claim that if $a_1b_2 = a_2b_1$, then $\vec{v}$ and $\vec{w}$ are parallel. Check this by cases:

(a) If $a_1 = 0$, then $a_1b_2 = 0$, so by assumption $a_2b_1 = 0$. Since $(a_1, a_2) = \vec{v} \neq (0, 0)$, and $a_1 = 0$, we must have $a_2 \neq 0$, and so $b_1 = 0$. In this case, $\vec{v} = (0, a_2)$ and $\vec{w} = (0, b_2)$ are parallel.

(b) If $a_2 = 0$, a similar argument shows $\vec{v}$ and $\vec{w}$ are parallel.

(c) If $a_1 \neq 0$ and $a_2 \neq 0$, we can divide $a_1b_2 = a_2b_1$ by a_1a_2 to get

$$\frac{b_2}{a_2} = \frac{b_1}{a_1} = d,$$

from which we have

$$d(a_1, a_2) = (da_1, da_2) = \left(\frac{b_1}{a_1}\, a_1, \frac{b_2}{a_2}\, a_2\right) = (b_1, b_2),$$

showing $\vec{v}$ and $\vec{w}$ are parallel.

This proves the claim.

Now, since $\vec{v}$ and $\vec{w}$ are not parallel, the claim tells us we must have $a_1b_2 \neq a_2b_1$, and so $a_1b_2 - a_2b_1 \neq 0$, In that case, we can check that

$$s = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad t = \frac{a_1c_2 - a_2c_1}{a_2b_2 - a_2b_1}$$

is a solution of the system of linear equations.

This is what we needed to show to complete the argument for Case 3. Therefore, this completes the proof.

Exercise: Prove that the following subsets of $\mathbb{R}^3$ are equal:

$$X = \{(a, b, c) \in \mathbb{R}^3 \mid a + c = b\};$$

$$Y = \{s(1, 1, 0) + t(0, 1, 1) \mid s \in \mathbb{R} \,\&\, t \in \mathbb{R}\}.$$

Note: Two sets are equal if and only if they have exactly the same elements. So, to prove two sets of vectors X and Y are equal, prove:

For every vector v,

$$v \in X \iff v \in Y.$$