Math 24
Winter 2014
Practice with Proofs: Some Solutions

Exercise 1: Let V be a vector space over F. Prove that for any scalars $a, b \in F$ and any vector $\vec{x} \in V$, if

$$a\vec{x} = b\vec{x} \quad \& \quad \vec{x} \neq \vec{0},$$

then $a = b$.

You may use the axioms for a vector space, and the theorems and corollaries stated in Section 1.1 of the textbook, namely the cancellation law for vector addition, the uniqueness of the additive identity of V, the uniqueness of the additive inverse of $\vec{v}$, and the identities
$0\vec{x} = \vec{0}$,
$(-a)\vec{x} = -(a\vec{x}) = a(-\vec{x})$,
$a\vec{0} = \vec{0}$.
You may also use anything you know about addition and multiplication of scalars.

Do NOT use subtraction of vectors. Subtraction is not a basic operation; $\vec{v} - \vec{w}$ is defined to mean $\vec{v} + (-\vec{w})$. Because our axioms and results are about vector addition and additive inverses, not (explicitly) about subtraction of vectors, you should write $\vec{v} + (-\vec{w})$ rather than $\vec{v} - \vec{w}$ in your proof.

Give a very clear and complete proof, in complete English sentences, with each step explained.

Write out your group's proof on a separate page. Make sure it will be clear and readable to another group. This is an exercise in proof *writing.* (Your group needs only one copy of your proof.)

Proposition: Let V be a vector space over F. For any scalars $a, b \in F$ and any vector $\vec{x} \in V$, if

$$a\vec{x} = b\vec{x} \quad \& \quad \vec{x} \neq \vec{0},$$

then $a = b$.

Proof: Suppose $a, b \in F$, $\vec{x} \in V$, $a\vec{x} = b\vec{x}$ and $\vec{x} \neq \vec{0}$. We must show that $a = b$.

To do this, we suppose that $a \neq b$ and derive a contradiction. Since $a \neq b$, we know $a - b \neq 0$, and so $a - b$ has a multiplicative inverse $(a-b)^{-1}$. We will use this fact.

We have

$$a\vec{x} = b\vec{x}.$$

By vector space axiom (VS4), $b\vec{x}$ has an additive inverse $-(b\vec{x})$. Adding this to both sides, we get

$$(a\vec{x}) + (-(b\vec{x})) = (b\vec{x}) + (-(b\vec{x})).$$

By the definition of additive inverse, $(b\vec{x}) + (-(b\vec{x})) = \vec{0}$, and by part (b) of Theorem 1.2, $-(b\vec{x}) = (-b)\vec{x}$. Substituting into the above equation we get

$$a\vec{x} + (-b)\vec{x} = \vec{0}.$$

By vector space axiom (VS 8), $a\vec{x} + (-b)\vec{x} = (a + (-b))\vec{x}$, and we know $a + (-b) = a - b$. Substituting into the above equation, we get

$$(a-b)\vec{x} = \vec{0}.$$

Multiplying on both sides by $(a-b)^{-1}$, we get

$$(a-b)^{-1}((a-b)\vec{x}) = (a-b)^{-1}\vec{0}.$$

By vector space axioms (VS 6) and (VS5),

$$(a-b)^{-1}((a-b)\vec{x}) = ((a-b)^{-1}(a-b))\vec{x} = 1\vec{x} = \vec{x},$$

and by part (c) of Theorem 1.2, $(a-b)^{-1}\vec{0} = \vec{0}$. Substituting into the previous equation, we get

$$\vec{x} = \vec{0}.$$

This contradicts our original supposition that $\vec{x} \neq \vec{0}$, which completes the proof. □

A slightly different proof:

Proposition: Let V be a vector space over F. For any scalars $a, b \in F$ and any vector $\vec{x} \in V$, if

$$a\vec{x} = b\vec{x} \quad \& \quad \vec{x} \neq \vec{0},$$

then $a = b$.

Before proving this, we will prove a useful lemma.

Lemma: Let V be a vector space over F, $c \in F$, and $\vec{x} \in V$. If $c\vec{x} = \vec{0}$, then either $c = 0$ or $\vec{x} = \vec{0}$.

Proof of Lemma: Suppose that $c\vec{x} = \vec{0}$ and $c \neq 0$. We must show that, in this case, $\vec{x} = \vec{0}$.

Since $c \neq 0$, we know c has a multiplicative inverse c^{-1}. We will use this fact to give a two-column proof that $\vec{x} = \vec{0}$:

$$\begin{aligned}
c\vec{x} &= \vec{0} && \text{given;} \\
c^{-1}(c\vec{x}) &= c^{-1}\vec{0} && \text{multiply both sides by } c^{-1}; \\
c^{-1}(c\vec{x}) &= \vec{0} && \text{Theorem 1.2 (c);} \\
(c^{-1}c)\vec{x} &= \vec{0} && \text{(VS6);} \\
(1)\vec{x} &= \vec{0} && \text{definition of } c^{-1}; \\
\vec{x} &= \vec{0} && \text{(VS5).}
\end{aligned}$$

This proves the lemma.

Proof of Proposition: Suppose that $a, b \in F$, $a\vec{x} = b\vec{x}$, and $\vec{x} \neq 0$. We must show that $a = b$. We begin with a two-column proof:

$$\begin{aligned}
a\vec{x} &= b\vec{x} && \text{given;} \\
(a\vec{x}) + (-(b\vec{x})) &= (b\vec{x}) + (-(b\vec{x})) && \text{add } -(b\vec{x}) \text{ to both sides;} \\
(a\vec{x}) + (-(b\vec{x})) &= \vec{0} && \text{definition of } -(b\vec{x}) \text{ (VS4);} \\
(a\vec{x}) + ((-b)\vec{x})) &= \vec{0} && \text{Theorem 1.2(b));} \\
(a + (-b))\vec{x} &= \vec{0} && \text{(VS8).}
\end{aligned}$$

Now, by the lemma, since $\vec{x} \neq \vec{0}$, from $(a+(-b))\vec{x} = \vec{0}$, we get $a+(-b) = 0$, or $a - b = 0$, or $a = b$. This completes the proof.

Exercise 2: We know that $M_{2\times 2}(F)$ is a vector space over F. Prove that

$$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F) \;\middle|\; c = 0 \right\},$$

with the same addition and scalar multiplication as in $M_{2\times 2}(F)$, namely

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} \quad \& \quad r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix},$$

is also a vector space over F.

You may note (and use in your proof) the fact that vector space axioms (VS1), (VS2), (VS5), (VS6), (VS7), and (VS8) automatically hold in V, because the operations of V are the same as the operations of $M_{2\times 2}(F)$, which we know is a vector space.

(If you have read Section 1.3, you will see that we are proving V is a *subspace* of $M_{2\times 2}(\mathbb{R})$. For this exercise, do not use the results of Section 1.3.)

Exercise 3: Prove that $M_{2\times 2}(\mathbb{R})$, with addition and scalar multiplication defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix},$$

$$r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix},$$

is *not* a vector space over $\mathbb{C}$.

After proving this, go back to Exercise 2 and check that your proof is complete.

Comment: For Exercise 3, we do not have a vector space over $\mathbb{C}$, because if we multiply an element of $M_{2\times 2}(\mathbb{R})$ by a complex number (for example, if we multiply $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by i), we do not always get an element of $M_{2\times 2}(\mathbb{R})$. Part of the definition of a vector space V over a field F is that, for every vector $v \in V$ and scalar $a \in F$, there is a unique vector av in V. Here our scalar multiplication does not always give an element of V, so V is not a vector space over $\mathbb{C}$.

Once you see this, you see that in Exercise 2, we need to prove that sums and scalar multiples of matrices in V are again elements of V. That is, we need to show V is closed under addition and mulltiplication by scalars.

Exercise 4:
Proposition: If V is any vector space over $\mathbb{R}$, and $\vec{x} \in V$ is a vector such that $\vec{x} + \vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.

Proof: Suppose $\vec{x} + \vec{x} = \vec{0}$. The following two-column proof shows that then $\vec{x} = \vec{0}$.

$\vec{x} + \vec{x} = \vec{0}$	given
$(1 \cdot \vec{x}) + (1 \cdot \vec{x}) = \vec{0}$	(VS 5)
$(1 + 1) \cdot \vec{x} = \vec{0}$	(VS 8)
$2 \cdot \vec{x} = \vec{0}$	arithmetic (2 means $1 + 1$)
$\frac{1}{2} \cdot (2 \cdot \vec{x}) = \frac{1}{2} \cdot \vec{0}$	$2 \neq 0$ so 2 has a multiplicative inverse
$\left(\frac{1}{2} \cdot 2\right) \cdot \vec{x} = \frac{1}{2} \cdot \vec{0}$	(VS6)
$1 \cdot \vec{x} = \frac{1}{2} \cdot \vec{0}$	definition of multiplicative inverse
$\vec{x} = \frac{1}{2} \cdot \vec{0}$	(VS5)
$\vec{x} = \vec{0}$	Theorem 1.2 (c)

Proposition: It is not always the case, for a vector space V over a field F, that if $\vec{x} \in V$ is a vector such that $\vec{x} + \vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.

Comment: To prove this, we need only find a counterexample; since this differs from the previous proposition only in having "a field F" in place of "$\mathbb{R}$," we look through the previous proof to see where we used any properties of $\mathbb{R}$ other than the axioms for a field.

Examining lines (4) and (5) of the two-column proof, we note that every field does have an element $1 + 1$, which we could call 2; but then, the claim "$2 \neq 0$" is really "$1 + 1 \neq 0$," which is not true in the field $\mathbb{Z}_2$. Let's try to find a counterexample using this field.

Let $F = \mathbb{Z}_2$, $V = F^2$, and $\vec{x} = (1, 1)$. Then, since in this vector space $\vec{0} = (0, 0)$ and $1 \neq 0$, we have $\vec{x} \neq \vec{0}$. On the other hand, since $1 + 1 = 0$ in $\mathbb{Z}_2$, we have

$$\vec{x} + \vec{x} = (1, 1) + (1, 1) = (1 + 1,\ 1 + 1) = (0, 0) = \vec{0},$$

which gives us the desired counterexample.

A slightly different proof:

Proposition: If V is any vector space over $\mathbb{R}$, and $\vec{x} \in V$ is a vector such that $\vec{x} + \vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.

Proof: Suppose $\vec{x}+\vec{x} = \vec{0}$ but $\vec{x} \neq \vec{0}$. The following two-column proof, using Exercise 1, leads to a contradiction.

$$\begin{array}{rr}
\vec{x} + \vec{x} = \vec{0} & \text{given} \\
(\vec{x} + \vec{x}) + (-\vec{x}) = \vec{0} + (-\vec{x}) & \text{(VS 4)} \\
\vec{x} + (\vec{x} + (-\vec{x})) = (-\vec{x}) + \vec{0} & \text{(VS 2) and (VS1)} \\
\vec{x} + \vec{0} = -\vec{x} & \text{(VS 4) and (VS3)} \\
\vec{x} = -(1 \cdot \vec{x}) & \text{(VS 3) and (VS5)} \\
1 \cdot \vec{x} = (-1) \cdot \vec{x} & \text{(VS5) and Theorem 1.2(b)} \\
1 = -1 & \text{Exercise 1}
\end{array}$$

Since $1 \neq -1$, we have our desired contradiction.

Proposition: It is not always the case, for a vector space V over a field F, that if $\vec{x} \in V$ is a vector such that $\vec{x} + \vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.

Comment: To prove this, we need only find a counterexample; since this differs from the previous proposition only in having "a field F" in place of "$\mathbb{R}$," we look through the previous proof to see where we used any special properties of $\mathbb{R}$. The two-column proof uses only axioms and theorems that apply to any vector space over any field. The only remaining step is the claim that $1 \neq -1$; that is, that 1 is not its own additive inverse, or, $1 + 1 \neq 0$.

This holds in our usual examples of fields. However, in the field $\mathbb{Z}_2$, we do have $1 + 1 = 0$. Let's try to find a counterexample using this field.

Let $F = \mathbb{Z}_2$, $V = F^2$, and $\vec{x} = (1, 1)$. Then, since in this vector space $\vec{0} = (0, 0)$ and $1 \neq 0$, we have $\vec{x} \neq \vec{0}$. On the other hand, since $1 + 1 = 0$ in $\mathbb{Z}_2$, we have

$$\vec{x} + \vec{x} = (1, 1) + (1, 1) = (1 + 1,\ 1 + 1) = (0, 0) = \vec{0},$$

which gives us the desired counterexample.