Mathematics 24
Winter 2014
Picturing Linear Transformations and Proving the Dimension Theorem

We'll begin by talking about visualizing linear transformations, and consequences of the dimension theorem. We'll talk about proving the dimension theorem later.

First, let's consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (2x + y,\ x + 2y).$$

We can check that the only solution to $T(x, y) = (0, 0)$ is $(x, y) = (0, 0)$, so the null space (or kernel) of T is $\{(0, 0)\}$, and the nullity of T is $n(T) = 0$. Because the nullity of T is 0, we know that T is one-to-one.

The dimension theorem tells us that the rank of T is given by

$$r(T) = dim(domain(T)) - n(T) = 2 - 0 = 2.$$

In this case, the range (or image) of T is a dimension 2 subspace of the codomain. Since the range of T has the same dimension as the codomain $\mathbb{R}^2$, we know that T is onto.

We can visualize T as follows: We can think of the standard basis of $\mathbb{R}^2$ as imposing a grid on $\mathbb{R}^2$, consisting of lines parallel to basis vector $(1, 0)$ and lines parallel to basis vector $(0, 1)$. The transformation T picks up these basis vectors and places them down again on $T(1, 0)$ and $T(0, 1)$ (in this case, $(2, 1)$ and $(1, 2)$), carrying the entire grid along.

The pictures on the next page illustrate this. Figure 1 shows part of the grid (including the edges of the unit square, the basis vectors $(1, 0)$ and $(0, 1)$, shown as thick lines of red and green, respectively). Figure 2 shows where this part of the grid is mapped by T. It's not hard to imagine how a point inside the unit square is carried along to a corresponding position inside the parallelogram that is the image of the unit square.

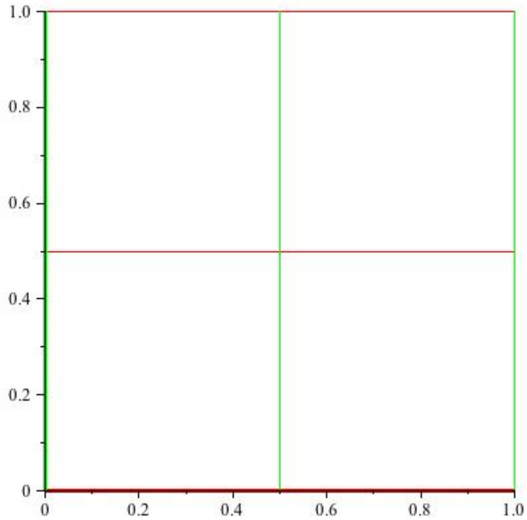

Figure 1: This is the unit square, with edges the standard basis vectors.

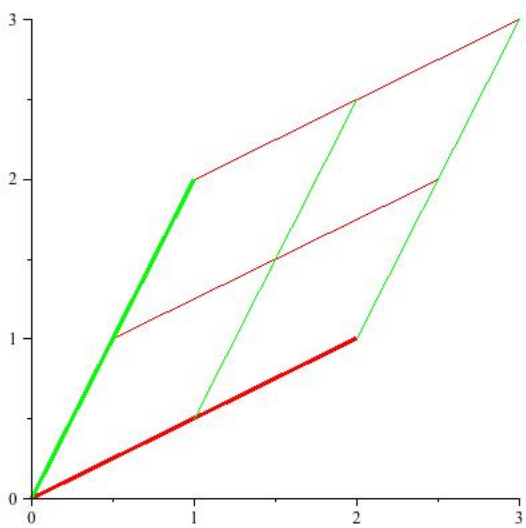

Figure 2: This is the image of the unit square.

Now let's consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x - y,\ x - z).$$

The general solution to $T(x, y, z) = (0, 0)$ is $(x, y, z) = (s, s, s)$, where s can be any real number. Therefore a basis for $N(T)$ is $\{(1, 1, 1)\}$, and the nullity of T is $n(T) = 1$. Since the nullity of T is not zero, we know T is not one-to-one.

The dimension theorem tells us that the rank of T is given by

$$r(T) = dim(domain(T)) - n(T) = 3 - 1 = 2.$$

In this case, the range (or image) of T is a dimension 2 subspace of the codomain. Since the range of T has the same dimension as the codomain $\mathbb{R}^2$, we know that T is onto.

We can try to use the same strategy as before to visualize T. Figure 3 on the next page pictures the edges of the unit cube in $\mathbb{R}^3$, with the standard basis vectors indicated by thick lines of red, green, and blue. Figure 4 shows where in the plane $\mathbb{R}^2$ this cube is mapped by T. Apparently T flattens the cube onto a portion of $\mathbb{R}^2$, but it's not as easy as in our last example to visualize what happens to a point in the interior.

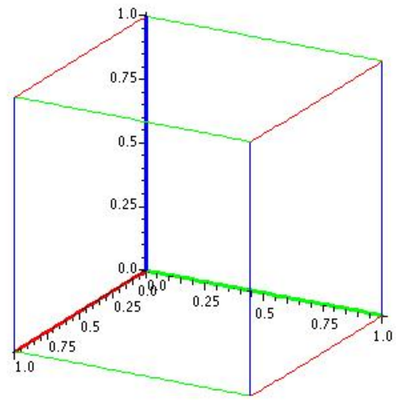

Figure 3: This is the unit cube in $\mathbb{R}^3$ with edges the standard basis vectors.

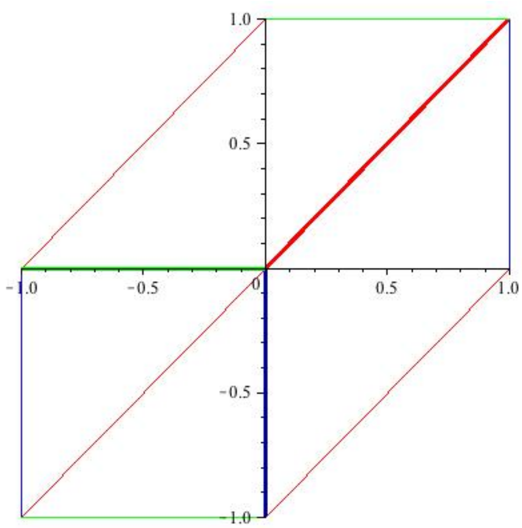

Figure 4: This is the image in $\mathbb{R}^2$ of the unit cube.

We are still considering the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x - y,\, x - z).$$

It will be easier to visualize T if we think about a different basis.

We saw that a basis for $N(T)$ is $\{(1, 1, 1\}$. We can extend this to a basis for $\mathbb{R}^3$, say $\{(1, 1, 1),\, (1, -1, 0),\, (0, 1, -1)\}$.

Figure 5 on the next page shows the edges of the parallelopiped with these vectors as edges, with the basis vectors again indicated by thick lines of red, green, and blue. Figure 6 shows where in the plane $\mathbb{R}^2$ this parallelopiped is mapped by T. This picture is cleaner, because the red lines all go to points. The parallelogram is the image of the three-dimensional parallelopiped, and it's also the image of the face of the parallelopiped that has the thick blue and green basis vectors as edges.

If we call that face F, we can visualize T as follows: First flatten the entire parallelopiped onto face F, by contracting the red edges to points. Points in the interior of the parallelopiped travel along lines parallel to the red edges to reach points on F. Now, send F to $\mathbb{R}^2$ by sending the thick green and blue basis vectors to the vectors shown in Figure 6, carrying points inside face F to corresponding positions inside the parallelogram that is the image of F.

Of course, T is actually defined on all of $\mathbb{R}^3$, but you can think of it the same way: First flatten all of $\mathbb{R}^3$ onto the plane containing F, by moving every point along a line parallel to the red basis vector. Then send the green and blue basis vectors to their images in $\mathbb{R}^2$, carrying the entire plane along.

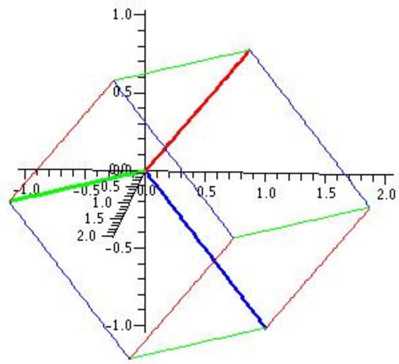

Figure 5: This is a parallelopiped, with edges our new basis vectors.

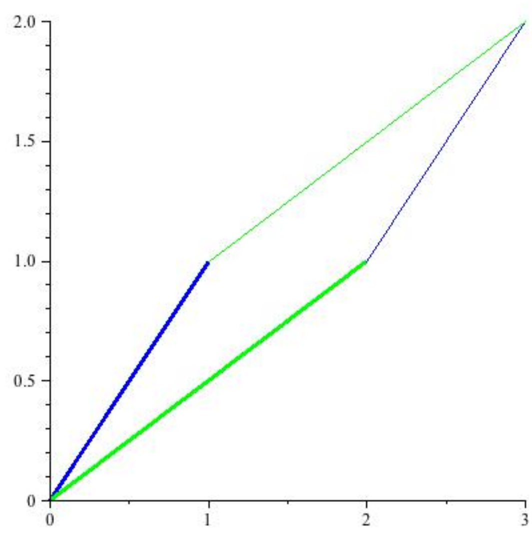

Figure 6: This is the image of the parallelopiped. The red lines go to points.

This strategy of extending a basis for $N(T)$ to a basis for the domain of T is used in the proof of the dimension theorem. The idea is that, as in our pictures, the elements of the basis of $N(T)$ go to zero, and the remaining basis elements go to non-interfering (linearly independent) images that form a basis for the range $R(T)$.

So suppose $T : V \to W$, where V has finite dimension m, and $N(T)$ has dimension $n \leq m$. Let $\{x_1, \dots, x_n\}$ be a basis for $N(T)$.

Since every linearly independent subset of V can be extended to a basis for V, and every basis for V contains exactly m elements, we can extend our basis for $N(T)$ to a basis $\{x_1, \dots, x_n, y_1, \dots, y_{m-n}\}$ for V.

We will show that $\{T(y_1), \dots, T(y_{m-n})\}$ is a basis for the range $R(T)$. Then we will have $r(T) = dim(R(T)) = m - n$. Since $m = dim(V)$ and $n = dim(N(T)) = n(T)$, this will give

$$r(T) + n(T) = dim(V),$$

which is the dimension theorem.

First, to show $\{T(y_1), \dots, T(y_{m-n})\}$ spans the range of T, we must show that every vector $T(v)$ in the range can be written as a linear combination of these elements.

Every $v \in V$ can be written as a linear combination of the elements of our basis for V,

$$v = a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_{m-n}y_{m-n}.$$

Applying T, and using its linearity and the fact that the x_i are in the null space of T, we see

$$\begin{aligned} T(v) &= T(a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_{m-n}y_{m-n}) \\ T(v) &= a_1T(x_1) + \cdots + a_nT(x_n) + b_1T(y_1) + \cdots + b_{m-n}T(y_{m-n}) \\ T(v) &= a_10 + \cdots + a_n0 + b_1T(y_1) + \cdots + b_{m-n}T(y_{m-n}) \\ T(v) &= b_1T(y_1) + \cdots + b_{m-n}T(y_{m-n}), \end{aligned}$$

which is what we needed to show.

Now, to show $\{T(y_1), \dots, T(y_{m-n})\}$ is linearly independent, suppose not, and deduce a contradiction. Because we are assuming this set is linearly dependent, we can express the zero vector as a nontrivial linear combination of the vectors in $\{T(y_1), \dots, T(y_{m-n})\}$,

$$0 = c_1T(y_1) + \cdots c_{m-n}T(y_{m-n}),$$

where at least one c_i is nonzero. We use the linearity properties of T again:

$$0 = T(c_1y_1 + \cdots c_{m-n}y_{m-n}).$$

That is to say, $c_1y_1 + \cdots + c_{m-n}y_{m-n}$ is an element of the null space of T. Now $\{x_1, \dots, x_n\}$ is a basis for $N(T)$, which means we can express any element of $N(T)$ as a linear combination of these vectors. In particular, we can write

$$c_1y_1 + \cdots + c_{m-n}y_{m-n} = d_1x_1 + \cdots + d_nx_n$$

$$c_1y_1 + \cdots + c_{m-n}y_{m-n} - d_1x_1 - \cdots d_nx_n = 0.$$

Since at least one c_i is nonzero, we have written the zero vector as a nontrivial linear combination of elements of the basis $\{x_1, \dots, x_n, y_1, \dots, y_{m-n}\}$. Since a basis is linearly independent, this is the contradiction we were looking for.