Math 24
Winter 2014
Some Solutions from Wednesday, January 22

First some definitions. If W_1 and W_2 are two subspaces of V, we define

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \ \& \ w_2 \in W_2\}.$$

In other words, $W_1 + W_2$ is the collection of all vectors you can get by adding an element of W_1 to an element of W_2. If $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$, then we say V is the *direct sum* of W_1 and W_2, and we write $V = W_1 \oplus W_2$.

1. Prove that $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2. (In other words, $W_1 + W_2$ is the span of $W_1 \cup W_2$.)

   To see $W_1 + W_2$ is a subspace, check closure under addition and under multiplication by scalars. Let $w_1 + w_2$ and $w_1' + w_2'$ be any elements of $W_1 + W_2$, where $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$, and let a be any scalar. Then, since W_1 and W_2 are closed under addition and under multiplication by scalars,

   $$(w_1 + w_2) + (w_1' + w_2') = (w_1 + w_1') + (w_2 + w_2') \in W_1 + W_2,$$

   $$a(w_1 + w_2) = aw_1 + aw_2 \in W_1 + W_2.$$

   Also, $W_1 \subseteq W_1+W_2$, since every $w_1 \in W_1$ can be written $w_1 = w_1+0 \in W_1+W_2$. For the same reason, $W_2 \subseteq W_1+W_2$. We have shown W_1+W_2 is a subspace containing both W_1 and W_2.

   Clearly every element $w_1 + w_2$ of $W_1 + W_2$ is in $span(W_1 + W_2)$ so $W_1 + W_2 \subseteq span(W_1 \cup W_2)$.

   To show $W_1 + W_2 = span(W_1 \cup W_2)$, it remains only to show that $span(W_1 \cup W_2) \subseteq W_1 + W_2$. But this must be true, because we have shown $W_1 + W_2$ is a subspace containing $W_1 \cup W_2$, and $span(W_1 \cup W_2)$ is the smallest such subspace.

   We could also show $span(W_1 \cup W_2) \subseteq W_1 + W_2$ directly. Let $w \in span(W_1 + W_2)$. We can write w as a linear combination of elements of $W_1 \cup W_2$,

   $$w = a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots b_mv_m,$$

where $u_i \in W_1$ and $v_j \in W_2$. But then, $a_1u_1 + \cdots + a_nu_n \in W_1$, and $b_1v_1 + \cdots + b_mv_m \in W_2$, and

$$w = (a_1u_1 + \cdots + a_nu_n) + (b_1v_1 + \cdots b_mv_m) \in W_1 + W_2,$$

so $span(W_1 \cup W_2) \subseteq W_1 + W_2$.

2. Give examples of pairs of subspaces W_1 and W_2 of $\mathbb{R}^3$, neither of which is contained in the other, such that:

   (a) $W_1 + W_2 \neq \mathbb{R}^3$. In your example, what is $W_1 + W_2$?

   (b) $W_1 + W_2 = \mathbb{R}^3$, but $\mathbb{R}^3$ is not the *direct* sum of W_1 and W_2. In your example, what is $W_1 \cap W_2$?

   (c) $\mathbb{R}^3$ is the direct sum of W_1 and W_2.

   This is a homework problem.

3. Suppose W_1 and W_2 are both subspaces of a finite-dimensional vector space V. Make a conjecture about the relationship among the dimensions of W_1, W_2, $W_1 \cap W_2$, and $W_1 + W_2$.

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

   Intuitively, we add up the number of dimensions in W_1 and W_2, and then subtract the number of dimensions in the overlap, because they were counted twice.

4. Express $M_{2\times 2}(\mathbb{C})$ as the direct sum of two nonzero subspaces.

   There are many possible answers. A straightforward one is:

$$W_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \right\} \qquad W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \;\middle|\; c, d \in \mathbb{C} \right\}$$

   A possibly more interesting solution is to let W_1 be the subspace of matrices with zero trace, and W_2 be the subspace of diagonal matrices whose two diagonal entries are equal. It's easy to see their intersection contains only the zero matrix. You can see that together they span the entire space by writing down simple bases for W_1 and W_2,

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

   and observing that from them you can generate all the standard basis elements.

5. Express $P(\mathbb{R})$ as the direct sum of two nonzero subspaces in two ways.

   (a) One of the subspaces has finite dimension.

   (b) Both of the subspaces are infinite-dimensional.

   This is a homework problem.

6. Prove the conjecture you made in problem (3). Hint: A basis $\{x_1, \ldots, x_k\}$ for $W_1 \cap W_2$ can be extended to a basis $\{x_1, \ldots, x_k, y_1, \ldots y_n\}$ for W_1. It can also be extended to a basis $\{x_1, \ldots, x_k, z_1, \ldots z_m\}$ for W_2. For homework, you might want to verify your conjecture by looking at problem 29(a) of section 1.6 of the textbook. Please make a conjecture yourself first, though.

   This is a challenging homework problem.

7. Every vector in $W_1 + W_2$ can be expressed as a sum, $w_1 + w_2$, of vectors $w_1 \in W_1$ and $w_2 \in W_2$. In what cases is this expression *unique*? Prove your answer is correct.

   This answer is unique just in case $W_1 \cap W_2 = \{0\}$; that is, just in case the sum $W_1 + W_2$ is a direct sum.

   To show this, first suppose $W_1 + W_2 \neq \{0\}$, and let w be a nonzero element of $W_1 \cap W_2$. Then w can be expressed as a sum of a vector from W_1 and a vector from W_2 in more than one way, namely as $w + 0$ and as $0 + w$.

   Conversely, suppose that $W_1 \cap W_2 = \{0\}$. We must show any vector $w \in W_1 + W_2$ can be expressed as a sum of a vector from W_1 and a vector from W_2 in only one way. To do this, suppose we have two such expressions $w = w_1 + w_2$ and $w = w_1' + w_2'$. We must show $w_1 = w_1'$ and $w_2 = w_2'$.

   We have $w_1 + w_2 = w_1' + w_2'$, which we can rewrite as $w_1 - w_1' = w_2' - w_2$. Thus $w_1 - w_1'$ is in both W_1 and W_2. The only vector in both W_1 and W_2 is 0, so $w_1 - w_1' = 0$, and $w_1 = w_1'$. The same argument shows that $w_2 = w_2'$.