Comments on row reduction and inverses
Math 24, Winter 2012

As we discussed in class, row reduction is good for more than just solving systems of linear equations – or rather, solving systems of linear equations are good for more than simply finding coefficients of some particular linear combination. There are a few steps.

Observation 1. Every row operation corresponds to a matrix product. To execute a row operation on M, simply perform that row operation to I_n to obtain a matrix E, and take the product EM. For example:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M$$

produces a matrix which agrees with M on rows 2 and 3, but has as its row 1 the sum of M's rows 1 and 2.

Observation 2. When using row reduction to solve a system of linear equations, the constants of the system are ignored in deciding what row operations to perform. Only the coefficients matter.

Observation 3. In the matrix product AB, A acts on each of B's columns individually, without interaction. That is, the i^{th} column of AB is the product of A with the i^{th} column of B.

Putting together 2 and 3, we see that multiple systems of linear equations, provided they have the same coefficients, may be solved simultaneously by augmenting the matrix of coefficients with multiple columns, one for each set of constants.

Observation 4. We may interpret any system of linear equations as a question of the image of a linear transformation: is the vector of constants in the image of the linear transformation given by the matrix of coefficients? We may also turn that around and apply row reduction to figure out the image of a linear transformation.

Observation 5. A linear transformation T from $\mathbb{R}^n$ to itself is an isomorphism if and only if the standard basis is in T's image.

Putting together 4 and 5, we see we can determine whether T is an isomorphism by solving the system of linear equations with T's matrix M_T as its coefficients and each standard basis vector of $\mathbb{R}^n$ in turn as the constants. From our previous observations it follows we can actually do this all at once: augment M_T by I_n and row-reduce. If you obtain something of the form $[I_n \; M]$ (first n columns are the identity, second n columns are where the identity was before row reducing), then you know every basis vector of $\mathbb{R}^n$ is in the image of T because every one of the systems of linear equations has a solution.

Finally, we recall Observation 1, combined with 3, to see that we have really taken a string of row operation matrices and multiplied them by M_T and the identity matrix simultaneously but separately. Since that string of matrices, multiplied together into the single matrix M, gave the identity when multiplied by M_T, it must be M_T's inverse. Multiplying by I_n just gives M back, so the second n columns of the reduced matrix $[I_n \; M]$ are the inverse of M_T. If M_T will not row reduce to the identity, there will be vectors outside its image, so it is not invertible.

Example.

The matrix M_T below has linearly independent columns, so T is an isomorphism and so has an inverse. We will use row reduction to find its inverse.

$$M_T = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc ccc} 3 & 2 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

swap first two rows
also multiply original second row by -1

$$\left[\begin{array}{ccc ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

multiply row 1 by -3 and add to 2

$$\left[\begin{array}{ccc ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

multiply row 2 by $1/2$
multiply (current) row 2 by -1 and add to row 3

$$\left[\begin{array}{ccc ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 3/2 & 1/2 & 3/2 & 0 \\ 0 & 0 & -1 & -1 & -3 & 1 \end{array}\right]$$

multiply row 3 by -1 and add to row 2
multiply row 3 by $3/2$ and add to row 2
multiply row 3 by -1

$$\left[\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 & -3 & 3/2 \\ 0 & 0 & 1 & 1 & 3 & -1 \end{array}\right]$$

The row reduction above shows that the inverse matrix to M_T is

$$M_T^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 3/2 \\ 1 & 3 & -1 \end{bmatrix}$$

Interpreted as a collection of three systems of linear equations with the same coefficients but different constants, it also shows that $T(1, -1, 1) = (1, 0, 0)$, $T(2, -3, 3) = (0, 1, 0)$, and $T(-1, \frac{3}{2}, -1) = (0, 0, 1)$.